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A note on Deligne's formula

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ABSTRACT

Let R denote a Noetherian ring and an ideal $J \subset R$ with $U = \operatorname{Spec} R \setminus V(J)$. For an R-module M there is an isomorphism $\Gamma(U, \tilde{M}) \cong \varinjlim \operatorname{Hom}_R(J^n, M)$ known as Deligne's formula (see [8, p. 217] and Deligne's Appendix in [7]). We extend the isomorphism for any R-module M in the non-Noetherian case of R and $J = (x_1, \ldots, x_k)$ a certain finitely generated ideal. Moreover, we recall a corresponding sheaf construction.

1. Introduction

Let R denote a commutative Noetherian ring, $J \subseteq R$ an ideal and $U = \operatorname{Spec} R \setminus V(J)$. For an R-module M and its associated sheaf $\tilde{M}$ on $X = \operatorname{Spec} R$ it is known that the sheaf cohomology $H^i(U, \tilde{M})$ and the Čech cohomology $\check{H}^i(U, \tilde{M})$ are isomorphic for all $i \in \mathbb{Z}$ (see e.g. [8, III, 4]). In the non-Noetherian case a corresponding result holds whenever J is generated by a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_k$ (for the definition see [13]) and a covering of U by $\operatorname{Spec} R \setminus V(x_i), i = 1, \ldots, k$. For the details we refer to [10] and the monograph [13], where it is worked out in the frame of Commutative Algebra.

In the case of a Noetherian ring it is well-known that the global transform $\mathcal{D}_J(M) \cong \varinjlim \operatorname{Hom}_R(J^n, M)$ is isomorphic to $\check{H}^0(U, \tilde{M}) \cong \Gamma(U, \tilde{M}), U = X \setminus V(J)$, known as Deligne's formula (see e.g. [7], [8], [3] and [15]). Moreover, we shall contribute with a variant of Deligne's formula in the non-Noetherian case (generalizing arguments of [7] and [15]) for some particular classes of ideals J generated by $\underline{x} = x_1, \ldots, x_k$, where we put $\check{D}^0_{\underline{x}}(M) = \check{H}^0(U, \tilde{M})$. For our purposes here let $H_i(\underline{x}^{(n)}; M), i \in \mathbb{Z}$, denote the i-th Koszul homology of M with respect to $\underline{x}^{(n)} = x_1^n, \ldots, x_k^n$ for an integer $n \geq 1$. We prove the following:

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Theorem 1.1. *Let $J = (x_1, \ldots, x_k)R$ denote a finitely generated ideal in a commutative ring R. For an R-module M there is a commutative diagram*

$$\begin{array}{ccc} & \mathcal{D}_J(M) & \\ {}^{\theta_M}\swarrow & & \searrow^{\rho_M} \\ \varprojlim_{x\in J} M_x & \xrightarrow{\quad\sigma_M\quad} & \check{D}^0_{\underline{x}}(M) \end{array}$$

with the following properties:

(1) *θ_M and ρ_M are injective and σ_M is an isomorphism.*
(2) *θ_M resp. ρ_M is an isomorphism if and only if $\check{H}^1_{\underline{x}}(\mathcal{D}_J(M)) = 0$ (see 2.1 (B) for $\check{H}^1_{\underline{x}}(\cdot)$).*
(3) *θ_M resp. ρ_M is an isomorphism for every M if and only if the inverse system $\{H_1(\underline{x}^{(n)}; R)\}_{n\geq 1}$ is pro-zero, i.e. for any n there is an $m \geq n$ such that $H_1(\underline{x}^{(m)}; R) \to H_1(\underline{x}^{(n)}; R)$ is zero.*

Note that, if M is a Noetherian module, then $\{H_i(\underline{x}^{(n)}; M)\}_{n\geq 1}$ is pro-zero for any system of elements $\underline{x}$ and $i > 0$ as easily seen (see also [10]). Note that ρ_M is in general not onto, even for an injective R-module M (see Example 4.3). The study of inverse systems that are pro-zero was initiated by Greenlees and May (see [6]) and Alonso Tarrío, Jeremías López, Lipman (see [1]). For a further discussion about their notation we refer to [11] and to [13]. See also 2.2 for the notion of weakly pro-regular sequences and related subjects. Moreover, we present a description of $\check{D}^0_{\underline{x}}(M)$ as the sheaf $\tilde{M}(U)$ (see Section 5 for the details).

In Section 2 we recall some results about Čech and Koszul complexes needed in order to describe some properties of sequences generalizing those of regular sequences. In Section 3 we derive the homomorphisms of the above diagram. To this end we recall some constructions known in the case of Noetherian rings and modify them in the general case, not available in this form before. This is necessary for the proof of the main results done in Section 4. We continue in Section 4 with a necessary and sufficient homological condition for ρ_M to become an isomorphism and an Example clarifying the necessary conditions in 1.1. In our notation we follow (with some minor differences) those of [13]. Moreover [13] is our basic reference.

2. Recalls about sequences

At the beginning let us fix some notation. In the following we shall use these notations without further reference. Let R denote a commutative ring and let M denote an R-module. For a sequence of elements $\underline{x} = x_1, \ldots, x_k$ and an integer $n > 0$ put $\underline{x}^{(n)} = x_1,^n \ldots, x_k^n$. Moreover, let $J = (x_1, \ldots, x_k)R$ the ideal generated by the sequence $\underline{x}$.

Notation 2.1.

(A) We denote by $K_{\cdot}(\underline{x}^{(n)}; M)$ the Koszul complex of M with respect to the sequence $\underline{x}^{(n)}$. For an integer i let $H_i(\underline{x}^{(n)}; M)$ denote the i-th Koszul homology. For two positive integers $m \geq n$ there are natural maps $K_{\cdot}(\underline{x}^{(m)}; M) \to K_{\cdot}(\underline{x}^{(n)}; M)$ that induces homomorphisms on the corresponding homology modules. For each integer i they induce inverse systems $\{H_i(\underline{x}^{(n)}; M)\}_{n\geq 1}$.
(B) Let $K^{\cdot}(\underline{x}^{(n)}; M)$ denote the Koszul co-complex and $H^i(\underline{x}^{(n)}; M)$ its cohomology. It is known that $\varinjlim K^{\cdot}(\underline{x}^{(n)}; M) \cong \check{C}_{\underline{x}}(M)$, the Čech complex with respect to $\underline{x}$ (see e.g. [13]). That is,

$$\check{C}_{\underline{x}}(M) : 0 \to M \xrightarrow{d^0} \oplus_{i=1}^r M_{x_i} \xrightarrow{d^1} \oplus_{1\leq i<j\leq k} M_{x_i x_j} \xrightarrow{d^2} \ldots \to M_{x_1\cdots x_k} \to 0.$$

We call $\check{H}^i_{\underline{x}}(M) = H^i(\check{C}_{\underline{x}}(M)), i \in \mathbb{Z}$, the Čech cohomology of M with respect to $\underline{x}$. Moreover, let $\check{D}_{\underline{x}}(M)$ denote the global Čech complex given by

$$\check{D}_{\underline{x}}(M) : 0 \to \oplus_{i=1}^r M_{x_i} \xrightarrow{d^1} \oplus_{1 \le i < j \le k} M_{x_i x_j} \xrightarrow{d^2} \ldots \to M_{x_1 \cdots x_k} \to 0$$

(see also [13, 6.1]). There is a short exact sequence $0 \to \check{D}_{\underline{x}}(M)[-1] \to \check{C}_{\underline{x}}(M) \to M \to 0$ of complexes that induces an exact sequence $0 \to \check{H}^0_{\underline{x}}(M) \to M \to \check{D}^0_{\underline{x}}(M) \to \check{H}^1_{\underline{x}}(M) \to 0$ of R-modules, where we abbreviate $\check{D}^0_{\underline{x}}(M) := H^0(\check{D}_{\underline{x}}(M))$.

(C) Moreover let $\mathcal{D}_J(M) = \varinjlim \operatorname{Hom}_R(J^n, M)$ be the ideal transform of M with respect to J. We refer to [3], [13, Chapter 12, Section 5] and to [14] for more details. There is a natural homomorphism $\tau_M : M \to \mathcal{D}_J(M)$ and a short exact sequence

$$0 \to \Gamma_J(M) \to M \xrightarrow{\tau_M} \mathcal{D}_J(M) \to H^1_J(M) \to 0,$$

where $\Gamma_J(\cdot)$ denotes the J-torsion functor and $H^i_J(\cdot)$ its right derived functors, the local cohomology functors.

In the next we shall summarize some properties of sequences $\underline{x} = x_1, \ldots, x_k$ partially needed in the sequel.

Definition 2.2.

(A) A sequence $\underline{x} = x_1, \ldots, x_k$ is called M-weakly pro-regular, if for all $i > 0$ the inverse system $\{H_i(\underline{x}^{(n)}; M)\}_{n \ge 0}$ is pro-zero, where $H_i(\underline{x}^{(m)}; M) \to H_i(\underline{x}^{(n)}; M), m \ge n$, denotes the natural map induced by the Koszul complexes. That is, for each integer n there is an integer $m \ge n$ such that the map $H_i(\underline{x}^m; M) \to H_i(\underline{x}^n; M)$ is zero. We call $\underline{x}$ weakly pro-regular if it is R-weakly pro-regular. The first study of weakly pro-regular sequences has been done in [10].

(B) In regard to [2, Sect. 9, 6, def. 2] we call a sequence $\underline{x}$ weakly secant if the inverse system $\{H_1(\underline{x}^{(n)}; R)\}_{n \ge 1}$ is pro-zero. Therefore, a weakly pro-regular sequence is weakly secant too. The converse does not hold. Moreover, if R is a Noetherian ring, then any sequence $\underline{x}$ is weakly pro-regular.

In the following we give a characterization when $\underline{x}$ is weakly secant. This follows by some adaptions of arguments given in [13, 7.3.3] and [12, Proposition 5.3]. For the ring R let $R[T]$ denote the polynomial module in one variable. It is a free R-module with basis $\mathbb{N}$.

Lemma 2.3. *Let $\underline{x} = x_1, \ldots, x_k$ denote a system of elements in R. Then the following conditions are equivalent:*

(i) *$\underline{x}$ is weakly secant.*
(ii) *$\{H_1(\underline{x}^{(n)}; F)\}_{n \ge 1}$ is pro-zero for any flat R-module F.*
(iii) *$\varinjlim H^1(\underline{x}^{(n)}; I) = 0$ for any injective R-module I.*
(iv) *$\check{H}^1_{\underline{x}}(I) = 0$ for any injective R-module I.*
(v) *$\varprojlim H_1(\underline{x}^{(n)}; R[T]) = \varprojlim^1 H_1(\underline{x}^{(n)}; R[T]) = 0$.*

Proof. The equivalence of the first four conditions is a particular case of [13, 7.3.3] or [10, Lemma 2.4] for $i = 1$, where weakly pro-regular sequences are characterized. Next we show (ii) $\Longrightarrow$ (v). For $F = R[T]$ we get that $\{H_1(\underline{x}^{(n)}; R[T])\}_{n \ge 1}$ is pro-zero and (v) follows (see e.g. [13, 1.2.4]). The implication (v) $\Longrightarrow$ (i) is true by [4] (see also the argument in the paper [12]). □

3. The homomorphisms

In the following we recall the homomorphisms of the diagram in the Introduction. Moreover we add a few comments. To this end we use the previous notation.

Proposition 3.1. *(ρ_M) We define*

$$\rho_M : \mathcal{D}_J(M) \to \check{D}^0_{\underline{x}}(M),\ \phi \mapsto (\phi_n(x_i^n)/x_i^n)_{i=1}^k,$$

where $\phi_n \in \operatorname{Hom}_R(J^n, M)$ is a representative of $\phi \in \varinjlim \operatorname{Hom}_R(J^n, M)$. Then ρ_M is injective.

Proof. It is immediate to see that $\rho_M(\phi)$ does not depend upon the representative ϕ_n. Moreover we have $\phi_n(x_i^n)/x_i^n = \phi_n(x_j^n)/x_j^n$ for all i, j. Therefore $\rho_M(\phi) \in \operatorname{Ker} d^1 = \check{D}^0_{\underline{x}}(M)$. Now let $\rho_M(\phi) = 0$ and therefore $\phi_n(x_i^n)/x_i^n = 0$ for all $i = 1, \dots, k$. That is $x_i^m \phi_n(x_i^n) = 0$ for some $m \geq 0$. Because of $J^{n+m+k} \subseteq \underline{x}^{n+m} R$ it follows that $\phi_n(J^{n+m+k}) = 0$. Whence $\phi_n|_{J^{n+m+k}} = 0$ and therefore $\phi = 0$. □

Construction 3.2. Let $J \subset R$ denote an ideal. For an R-module M we look at the system of localizations $\{M_x\}_{x \in J}$. For elements $x, y \in J$ we define a partial order $y \geq x$ whenever $x \in \operatorname{Rad} yR$, i.e., $x^k = yr$ for some $r \in R$. Then we define a homomorphism $\alpha_{x,y} : M_y \to M_x$ by

$$\alpha_{x,y} : M_y \to M_x,\ m/y^n \mapsto r^n m/x^{kn}.$$

This is well-defined as easily seen. Moreover, if $x \in \operatorname{Rad} yR$ and $y \in \operatorname{Rad} zR$, then it follows that $\alpha_{x,y} \cdot \alpha_{y,z} = \alpha_{x,z}$ and $\alpha_{x,x} = \operatorname{id}_{M_x}$. So that $\{M_x\}_{x \in J}$ with the homomorphisms $\alpha_{x,y}$ forms an inverse system. Moreover, let $\alpha_x : \varprojlim_{x \in J} M_x \to M_x$ denote the canonical map. Note that for fractions m/x_i^a and n/x_j^b we use often the same exponent for the denominator.

Proposition 3.3. *(θ_M) We define*

$$\theta_M : \mathcal{D}_J(M) \to \varprojlim_{x \in J} M_x,\ \phi \mapsto (\phi_n(x^n)/x^n)_{x \in J},$$

where $\phi_n \in \operatorname{Hom}_R(J^n, M)$ is a representative of $\phi \in \varinjlim \operatorname{Hom}_R(J^n, M)$. Then θ_M is injective.

Proof. For any $x \in J$ and $\phi \in \mathcal{D}_J(M)$ let $\phi_n \in \operatorname{Hom}_R(J^n, M)$ be a representative of ϕ. Then we define $\phi \mapsto \phi_n(x^n)/x^n \in M_x$, which is well defined. This is compatible with the map $\alpha_{x,y} : M_y \to M_x$ with $x \in \operatorname{Rad} yR$, say $x^k = yr$ as easily seen. By the universal property of inverse limits there is a homomorphism $\theta_M : \mathcal{D}_J(M) \to \varprojlim_{x \in J} M_x$. The injectivity of θ_M follows as in the proof of 3.1. □

Now we are going on to construct the final morphism in the diagram of 1.1.

Proposition 3.4. *(σ_M) By the above notation let $\alpha_x : \varprojlim_{x \in J} M_x \to M_x$ the canonical map. We define*

$$\sigma_M : \varprojlim_{x \in J} M_x \to \check{D}^0_{\underline{x}}(M),\ f \mapsto (m_i/x_i^n)_{i=0}^k,\ \text{where } \alpha_{x_i}(f) = m_i/x_i^n,\ i = 1, \dots, k.$$

Moreover σ_M is an isomorphism and $\rho_M = \sigma_M \circ \theta_M$.

Proof. First note that $\alpha_{x_i x_j, x_i}(\alpha_{x_i}(f)) = \alpha_{x_i x_j, x_j}(\alpha_{x_j}(f)) = \alpha_{x_i x_j}(f)$ for all pairs i, j. This yields that $m_i/x_i^n = (x_j^n m_i)/(x_i x_j)^n = (x_i^n m_j)/(x_i x_j)^n = m_j/x_j^n$ and $\sigma_M(f) \in \check{D}^0_{\underline{x}}(M)$. Let $\sigma_M(f) = 0$ for some $f \in \varprojlim M_x$ and therefore $\alpha_{x_i}(f) = m_i/x_i^n = 0$ for $i = 1, \dots, k$. Let $x \in J$ and $\alpha_x(f) = m/x^n$, where n can be chosen independently of i, j and x. Since $x^n m_i/(xx_i)^n = x_i^n m/(xx_i)^n$ and $m_i/x_i^n = 0$, it follows that $\alpha_x(f) = m/x^n = 0$ for all $x \in J$ and $f = 0$, so σ_M is injective.

Now let $(m_i/x_i^n)_{i=1}^k \in \check{D}^0_{\underline{x}}(M) = \operatorname{Ker} d^1$ and therefore $m_i/x_i^n = m_j/x_j^n$ for all i, j. That is, $(x_i x_j)^c x_j^n m_i = (x_i x_j)^c x_i^n m_j$ for some integer c and $x_j^{c+n} m_i' = x_i^{c+n} m_j'$ with $m_l' = x_l^c m_l$ and $m_l/x_l^n = m_l'/x_l^{c+n}, l = 1, \ldots, k$. Now choose an element $y \in J$ and an integer d such that $y^d = \sum_{i=1}^k r_i x_i^{c+n}$ since $y \in \operatorname{Rad}(x_1^{c+n}, \ldots, x_k^{c+n})$. We define $m_y = \sum_{j=1}^k r_j m_j'$. Then

$$x_i^{c+n} m_y = \sum\nolimits_{j=1}^k x_i^{c+n} r_j m_j' = \sum\nolimits_{j=1}^k r_j x_j^{c+n} m_i' = y^d m_i'$$

and therefore $m_y/y^d = m_i/x_i^n$ for $i = 1, \ldots, k$. In order to show that $(m_y/y^d)_{y \in J}$ defines an element $f \in \varprojlim_{x \in J} M_x$ such that $\sigma_M(f) = (m_i/x_i^n)_{i=1}^k$ let $z \in J$ with $z^f = yr$, where m_z/z^e is chosen as before. By the equation above $x_i^{c+n} r^d m_y = z^{fd} m_i'$ and therefore $r^d m_y/z^{df} = m_i/x_i^n = m_z/z^e$ for $i = 1, \ldots, k$. That is, $\alpha_{z,y}(m_y/y^d) = m_z/z^e$, whence f is an element of the inverse limit. The final claim is obvious. $\square$

Question 3.5. Fix the previous notation. What is a necessary and sufficient condition on the sequence $\underline{x} = x_1, \ldots, x_k$ such that the homomorphisms

$$\rho_M : \mathcal{D}_J(M) \to \check{D}^0_{\underline{x}}(M) \text{ and } \theta_M : \mathcal{D}_J(M) \to \varprojlim{}_{x \in J} M_x,\ \phi \mapsto (\phi_n(x^n)/x^n)_{x \in J}$$

in 3.1 and in 3.3 become isomorphisms for an R-module M? For an answer see 4.2 below.

4. Proof of the main results

In the following we shall discuss when the R-homomorphism $\rho_M : \mathcal{D}_J(M) \to \check{D}^0_{\underline{x}}(M)$ is onto. To this end we need a technical result.

Lemma 4.1. *Let $\underline{x}$ and $J = \underline{x}R$ be as above. Let M denote an R-module. There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_J(M) & \longrightarrow & M & \longrightarrow & \mathcal{D}_J(M) & \longrightarrow & H^1_J(M) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow{\scriptstyle \rho_M} & & \downarrow & & \\
0 & \longrightarrow & \check{H}^0_{\underline{x}}(M) & \longrightarrow & M & \longrightarrow & \check{D}^0_{\underline{x}}(M) & \longrightarrow & \check{H}^1_{\underline{x}}(M) & \longrightarrow & 0.
\end{array}$$

The natural map $H^1_J(M) \to \check{H}^1_{\underline{x}}(M)$ is injective and $\check{D}^0_{\underline{x}}(M)/\mathcal{D}_J(M) \cong \check{H}^1_{\underline{x}}(M)/H^1_J(M)$. Moreover, ρ_M is an isomorphism if $\underline{x}$ is weakly pro-regular.

Proof. For the exact sequence at the bottom of the diagram see 2.1 (B). The exact sequence at the top is shown in 2.1 (C). The commutativity of the diagram is obvious. Because the third vertical map $\mathcal{D}_J(M) \to \check{D}^0_{\underline{x}}(M)$ is injective (see 3.1), so is the fourth one. The statement follows now. Finally if $\underline{x}$ is weakly pro-regular, then $H^1_J(M) \cong \check{H}^1_{\underline{x}}(M)$ (see e.g. [13, 7.4.5]). $\square$

It could be of some interest to describe a necessary and sufficient condition for ρ_M to become an isomorphism in terms of $\underline{x}$ and the R-module M.

Proposition 4.2. *With the previous notation there is a short exact sequence*

$$0 \to \mathcal{D}_J(M) \xrightarrow{\rho_M} \check{D}^0_{\underline{x}}(M) \to \check{H}^1_{\underline{x}}(\mathcal{D}_J(M)) \to 0$$

and an isomorphism $\check{H}^1_{\underline{x}}(M)/H^1_J(M) \cong \check{H}^1_{\underline{x}}(\mathcal{D}_J(M))$. *Moreover* ρ_M *is an isomorphism if and only if* $\check{H}^1_{\underline{x}}(\mathcal{D}_J(M)) = 0$.

Proof. Localizing the exact sequence of 2.1 (C) at $x \in J$ yields $M_x \cong \mathcal{D}_J(M)_x$ and therefore

$$\check{D}^0_{\underline{x}}(\mathcal{D}_J(M)) \cong \varprojlim{}_{x\in J} \mathcal{D}_J(M)_x \cong \varprojlim{}_{x\in J} M_x \cong \check{D}^0_{\underline{x}}(M)$$

(see 3.4). By the exact sequence in 2.1 (B) for the ideal transform $\mathcal{D}_J(M)$ it yields the short exact sequence of the statement. □

Proof of Theorem 1.1.
(1): These statements are shown in Propositions 3.1, 3.3 and 3.4.
(2): We have that ρ_M is an isomorphism if and only if θ_M is so. By 4.2 ρ_M is an isomorphism if and only if $\check{H}^1_{\underline{x}}(\mathcal{D}_J(M)) = 0$.
(3): If ρ_M is isomorphisms for any R-module it holds in particular for any injective R-module. In the diagram of 4.1 we have the vanishing $H^1_J(I) = 0$ for any injective R-module I. That is, $\rho_I : \mathcal{D}_J(I) \to \check{D}^0_{\underline{x}}(I)$ is an isomorphism for every injective R-module I if and only if $\check{H}^1_{\underline{x}}(I) = 0$ for every injective R-module I. By 2.3 this holds if and only if $\underline{x}$ is a weakly secant sequence. Now we prove that ρ_M is an isomorphism too. Let $0 \to M \to I^0 \to I^1$ be the beginning part of an injective resolution of M. It induces a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{D}_J(M) & \longrightarrow & \mathcal{D}_J(I^0) & \longrightarrow & \mathcal{D}_J(I^1) \\
& \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_{I^0}} & & \downarrow{\scriptstyle \theta_{I^1}} \\
0 \longrightarrow & \varprojlim_{x\in J} M_x & \longrightarrow & \varprojlim_{x\in J} (I^0)_x & \longrightarrow & \varprojlim_{x\in J} (I^1)_x .
\end{array}$$

For the first exact sequence recall $\mathcal{D}_J(M) = \ker(\mathcal{D}_J(I^0) \to \mathcal{D}_J(I^1))$. For the second note that localization is exact and passing to the inverse limit left exact. If $\underline{x}$ is weakly secant, then $\theta_{I^i}, i = 0, 1$, is an isomorphism and θ_M too. □

We conclude with an explicit example of a ring and an injective module such that ρ is not an isomorphism.

Example 4.3. (see also [14, Example 5.5]) Let $R = \Bbbk[[x]]$ denote the power series ring in one variable over the field $\Bbbk$. Let $E = E_R(\Bbbk)$ denote the injective hull of the residue field. Then define $S = R \ltimes E$, the idealization of R by the R-module E as introduced by M. Nagata (see [9]). That is, $S = R \oplus E$ as an R-module with a multiplication on S defined by $(r, r) \cdot (r', e') = (rr', re' + r'e)$ for all $r, r' \in R$ and $e, e' \in E$. By a result of Faith [5] we have that the commutative ring S is self-injective. More precisely, there is an isomorphism of S-modules $\operatorname{Hom}_R(S, E) \cong S$ (see also [13, Theorem A.4.6]). We consider the ideal $J := (x, 0)S$ of S and note $\Gamma_J(S) = 0 \ltimes E$. Then $\Gamma_J(S)$ is not injective as an S-module (see [13, 2.8.8]). We have that S is self-injective with $\Gamma_J(S) = 0 \ltimes E$. Hence $\mathcal{D}_J(S) = R \subsetneq \check{D}^0_{(x,0)}(S) = S_{(x,0)} = R_x$.

In this example the ascending sequence of ideals $0 :_S (x, 0)^t = 0 \ltimes (0 :_E x^t), t > 0$ does not stabilize. That is, S is not of bounded $(x, 0)$-torsion and therefore the inverse system $\{H_1((x, 0)^n; S)\}_{n\geq 1}$ is not pro-zero.

5. The associated sheaf

In this part of the paper we shall recall the sheaf construction of $\tilde{M}(U)$ for a finitely generated ideal $J = (x_1, \ldots, x_k)R$ in a commutative ring R and an R-module M. Note that it is closely related to the cohomological investigations. Set $X = \operatorname{Spec} R$ and $U = X \setminus V(J)$. Note that $D(x_i) = X \setminus V(x_i), i = 1, \ldots, k$, is an open covering of U.

Definition and observation 5.1. For the R-module M we consider the set $\{M_x\}_{x\in J}$. Because we are interested in the localization $M_x, x \in J$, we may replace M by $M/0 :_M \langle J\rangle$, where $0 :_M \langle J\rangle = \cup_{n\geq 1} 0 :_M J^n$. Note that $M_x = M_y$ for $x, y \in J$ with $\operatorname{Rad} xR = \operatorname{Rad} yR$. For elements $x, y \in J$ with $x \in \operatorname{Rad} yR$, i.e., $x^k = yr$ for some $r \in R$ we have as above a homomorphism $\alpha_{x,y} : M_y \to M_x$ (see 3.2). Now recall that $\{D(x_i)|i = 1, \dots, k\}$, is an affine covering of U.

Claim. *$\tilde{M}(U) := (\{M_x\}_{x\in J}, \{\alpha_{x,y}| \operatorname{Rad} xR \subseteq \operatorname{Rad} yR\}) = (\{M_x\}_{x\in J}, \{\alpha_{x,y}|D(x) \subseteq D(y)\})$ is a sheaf of modules.*

Proof. To this end we have to show the following:

(1) If $m/x^n \in M_x$ maps to zero in R_{x_i} for all $i = 1, \dots, k$, then $m = 0$.
(2) If $m_j/x_j^n \in M_{x_j}, j = 1, \dots, k$, satisfies $m_i/x_i^n = m_j/x_j^n$ in $M_{x_ix_j}$ for all i, j. Then there is an $n/y \in M_y$ that maps to m_i/x_i^n for all $i = 1, \dots, k$.

If m/x^n maps to zero in R_{x_i}, then $x_i^c m = 0$ for all $i = 1, \dots, k$, where c can be chosen independently of i. That is $J^{ck}m = 0$ and $m = 0$ and (1) holds. In order to show (2) note that we may chose n independently of i, j. Then the assumption implies $(x_ix_j)^c x_j^n m_i = (x_ix_j)^c x_i^n m_j$ for a certain c for all i, j. We put $m_i' = x_i^c m_i$ and get $m_i/x_i^n = m_i'/x_i^{c+n}$ and $x_j^{c+n} m_i' = x_i^{c+n} m_j'$ for all i, j. Now we choose $y = \sum_{i=1}^k r_i x_i^{c+n}$ and $m = \sum_{j=1}^k r_j m_j'$. Then

$$x_i^{c+n} m = \sum\nolimits_{j=1}^k r_j x_i^{c+n} m_j' = \sum\nolimits_{j=1}^k r_j x_j^{c+n} m_i' = y m_i'$$

and $m/y = m_i'/x_i^{c+n} = m_i/x_i^n$ for all $i = 1, \dots, k$.

Moreover, clearly $\tilde{M}(U)$ is a sheaf of $\tilde{A}(U)$-modules. We conclude with the obvious remark that $\tilde{M}(U)$ coincides with $\check{D}^0_{\underline{x}}(M)$ for a finitely generated ideal.

Corollary 5.2. *For an R-module M and an ideal $J \subset R$ we have $\check{D}^0_{\underline{x}}(M) \cong \tilde{M}(U)$, where $\underline{x} = x_1, \dots, x_k, J = (x_1, \dots, x_k)R$ and $U = X \setminus V(J)$.*

Proof. Let $f = (m_i/x_i^n)_{i=1}^k \in \check{D}^0_{\underline{x}}(M)$ and therefore $m_i/x_i^n = m_j/x_j^n$ for all i, j. By virtue of (2) in 5.1 there is an $n/y \in M_y$ that maps to m_i/x_i^n for all $i = 1, \dots, k$. Whenever, n/y maps to zero by the map of 5.1, the condition (1) in 5.1 implies that $n = 0$. So, there is an isomorphism $\check{D}^0_{\underline{x}}(M) \cong \tilde{M}(U)$. □

CRediT authorship contribution statement

Peter Schenzel: Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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