# On Construction of Piecewise Constant Orthonormal Functions Based on Rescaling Cantor Set with Their Application in Orthogonal Multiplexing Systems

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Abstract: The purpose of this paper is to construct a novel system of discontinuous piecewise constant orthogonal functions that is complete with respect to the measure on 4-adic-type Cantor-like sets, particularly on a rescaled Cantor set. The construction process is rigorously developed, and an accurate method for generating these functions is presented. This orthogonal function system is then applied within the framework of an orthogonal multiplexing scheme, providing a practical solution for communication systems. A numerical example illustrates the use of the proposed system as a communication carrier signal designed to reduce multiple access interference in communication channels. The input signal is approximated using these piecewise constant functions, which are naturally computed through a specific Fourier series expansion. Following a formal introduction of this Fourier series, the procedure for obtaining the corresponding Fourier coefficients of the input signal is detailed. These coefficients are then transmitted through the designed multiplexing system to enable efficient and interference-free communication.

## 1 INTRODUCTION

In 1875, the British mathematician Henery J.S. Smith [24] was first defined the Cantor set which was studied and first published by the German mathematician Geroge Cantor, in 1883 [25]. It was an important example of a perfect nowhere dense set in the real line. There were many authors interested in construction of Cantor set such as : Robert D. and Wilfredo O. discussed several variations and generalizations of the Cantor set and studied some of their properties [26]. Also, for each of those generalizations a Cantor-like function can be constructed from the set. They discussed briefly the possible construction of those functions. Alireza Khalili Golmankhaneh, Arran Fernandez, Ali Khalili Golmankhaneh and Dumitru Baleanu [27] had generalized the $C^{\xi}$ -calculus on the generalized Cantor sets known as middle ξ- Cantor sets. They had suggested a calculus on the middle ξ- Cantor sets for different values of ξ with $0 < \xi < 1$. In the problem of approximation in the Cantor set [28] showed that the behavior when they consider dyadic approximation in the Cantor set was substantially different to considering triadic approximation in the Cantor set. The orthogonal system consists of three kinds of systems: first, the sinusoidal (Fourier-Hartley) system, second, the nonsinusoidal (piecewise constant namely Haar, Walsh, and Block-pulse) system, third, the class of orthogonal polynomials such as: Legendre, Hermite, Laguerre, Jacobi, Tchebcheff (first and second kinds), and Gegenbauer which are very important to study. Historically, the beginnings of the discontinuous piecewise constant orthogonal functions began when the Hungarian mathematician, Alfred Haar suggested a system of orthogonal functions in 1910 [16]. In 1923, the American mathematician J.L. Walsh published an article entitled "A closed set of normal orthogonal functions" [17] in which he described a system of complete orthonormal functions over the normalized interval [0,1), each function taking only the values +1 or -1, except at a finite number of discontinuity points, where take the value 0. Block-Pulse functions were introduced by Harmuth [19], when he used the Walsh-Fourier series in the communication system. Each function takes one value +1 in each subinterval of [0,1] and otherwise takes the value 0 as well as these functions formed an orthonormal system on

[0,1] but many authors proved Block-Pulse system was incomplete in Hilbert space $L^2[0,1]$ [18].

Many of the advantages of Walsh, Haar and Block-Pulse systems in many real life problems such as: series representation [1]-[6], spectroscopy [7]-[9], speech processing [10]-[12], multiplexing system [13]-[15], and other applications.

The proposed system in this paper belongs to piecewise constant orthogonal systems. A different complete orthonormal system of discontinuous piecewise with them proves on 4 –adic-type Cantor – like sets is considered.

## 2 SOME BASIC DEFINITIONS AND CONCEPTS

In this part, we give some basic definitions and concepts which are related to the complete orthonormal system on [0,1] and a vector space over the Gailos field $\mathbb{G} = \{0,1\}$ in linear block codes.

Definition 2.1 [22]: A system of functions $\{\mathcal{V}_i(x)\}_{i\in\Delta}$ (where $\Delta$ is finite or infinite or infinite countable set) is orthogonal on (0,1) if

$$\int_0^1 \mathcal{V}_{\mathrm{i}}(x)\mathcal{V}_{\mathrm{j}}(x)\mathrm{d}x = 0,\ \mathrm{i} \neq \mathrm{j},\ \forall \mathrm{i},\mathrm{j} \in \Delta.$$

Definition 2.2 [23]: A system of functions $\{\mathcal{V}_i(x)\}_{i\in\Delta}$is called an orthonormal system if

$$\int_0^1 \mathcal{V}_i(x)\mathcal{V}_j(x)dx = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases} for\ all\ i,j \in \Delta.$$

Definition 2.3 [23]: An orthonormal system of functions $\{\mathcal{V}_i(x)\}_{i=0}^{\infty}$ is complete on (0,1) if

$$\int_0^1 \mathcal{V}(x)\mathcal{V}_i(x)dx = 0,\ \forall i \text{ implies } \mathcal{V} \equiv 0.$$

Lemma 2.1: An orthonormal system $\{\mathcal{V}_i(x)\}_{i\in\Delta}$ is linearly independent on (0,1).

Definition 2.4 [19]: Consider an ordered sequence of $\alpha$-symbols as a vector with $\alpha$-components: $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_\gamma\}$, $\alpha_r \in \mathbb{G}$ for each $r = 1,2,\cdots,\gamma$, where $\gamma$ is a positive integer.

Definition 2.5 [18]: The sequence

$$\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_\gamma\},$$

is an $\gamma$-tuple over $\mathbb{G}$. The set of all $2^\gamma$ possible $\gamma$-tuples is called a vector space over $\{0,1\}$and we can denote it as $\mathfrak{F}_2^\gamma = \mathbb{G}^\gamma$.

Definition 2.6 [18]: The addition and inner product of a binary $\gamma$-tuple via a symbol over $\mathbb{G}$ are introduced as follows:

$$m \oplus p = \{m_1, m_2, \cdots, m_\gamma\} \oplus \{p_1, p_2, \cdots, p_\gamma\} = \\ = \{m_1 \oplus p_1, m_2 \oplus p_2, \cdots, m_\gamma \oplus p_\gamma\}$$

$$m \odot p = \{m_1, m_2, \cdots, m_\gamma\} \odot \{p_1, p_2, \cdots, p_\gamma\}$$

$$= (m_1 \odot p_1) \oplus (m_2 \odot p_2) \oplus \cdots \oplus (m_\gamma \odot p_\gamma)$$

$$= \sum\nolimits_{\oplus r=1}^{\gamma} (m_r \odot p_r).$$

Definition 2.7 [18]: for each $r \geq n-1, n \in \mathrm{N}$, we write the binary of $r$ as:

$$w_r = \{r_\gamma, r_{\gamma-1}, \cdots, r_1, r_0\} \equiv \sum_{d=0}^{\gamma} 2^d r_d,$$

where, $r_d \in \mathbb{G}$, $\forall d: d = 0,1,\cdots,\gamma$, then the Gray code representation of $r$ is defined via:

$$w_r^G = \{r_\gamma \oplus r_{\gamma+1}, r_{\gamma-1} \oplus r_\gamma, \cdots, r_1 \oplus r_2, r_0 \oplus r_1\},$$

where, $r_{\gamma+1} = 0$.

## 3 RESCALING CANTOR SET (R.C.S.) CONSTRUCTION

R.C.S. is one of the most important kinds and generalizations of Cantor ternary set. It was first introduced by Tsang K.Y. [21], when he used them as the analytical method to compute the dimensionality of strange attractors in two dimensional maps. His construction can be explained via the following relation:

Rescaling Cantor set is the sub set $\mathcal{K}$ of $\mathcal{R}$ given via:

$$\mathcal{K} = \bigcap_{r=0}^{\infty} \mathcal{K}_r,$$

where, $\mathcal{K}_0, \mathcal{K}_1, \cdots$ are computed via the following processing: begin with the closed interval $\mathcal{K}_0 = [0,1]$, and $\gamma \in N$. The set $\mathcal{K}_1$ is constructed from $\mathcal{K}_0$ via removing $\gamma - 1$ open intervals so the $\gamma$- closed intervals each of length $\mathcal{S}_s$ greater than zero ($s = 1,2,\cdots,\gamma$) of the interval remain. Another set $\mathcal{K}_2$ is constructed via repeating the above processing with each of the $k$- intervals in $\mathcal{K}_1$.

Definition (3.1): Let $k \in N$, with $k > 2$ and $1 < \gamma < k$. If $\mathcal{S}_s = 1/k$, for each $s = 1,2,\cdots,\gamma$, then the set $\mathcal{K}$ is called the $k$- adic- type Cantor-like set [21].

In order to reduce the effort required to construct a R.C.S., we give a new method to represent it in the case $k = 4$. This method is illustrated in the following formation:

For each $v \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}$, generating the closed intervals:

$$\mathcal{K}_\ell^4 = [\tfrac{3e}{4^v}, \tfrac{3e+1}{4^v}],\ 0 \le c < 2^v,$$

where, $\ell = 2^v + c$, and $e = \sum_{t=0}^{\infty} 4^t e_t$, where, $e_t$ being the dyadic coefficients of $c$ in binary system: $c = \sum_{t=0}^{\infty} 2^t c_t$ , $c_t \in \mathbb{G}$. It can be shown that for any integer $\ell$ there exists a unique integer $v$ and a unique integer $c, c \in \{0,1,\cdots,2^v - 1\}$ such that $\ell = 2^v + c$. Then the 4-adic-type Cantor-like set is :

$$\mathcal{K}^4 = \bigcap_{v=0}^{\infty} \bigcup_{c=0}^{2^v - 1} \mathcal{K}_\ell^4,$$

while the middle open intervals deleted in the above construction are introduced as:

$$\mathcal{Z}^4 = \bigcup_{v=0}^{\infty} \bigcup_{c=0}^{2^v - 1} \mathcal{Z}_\ell^4,$$

where,

$$\mathcal{Z}_\ell^4 = \left(\frac{12e+1}{4^{v+1}}, \frac{12e+3}{4^{v+1}}\right).$$

With the measure of $\mathcal{K}_\ell^4$ is equal to $2^{-v}$. Table 1 shows the first 15-closed intervals with their 7-open intervals of 4-adic-type Cantor-like sets.

The set $\mathcal{K}^4$ can be associated with a monotone non decreasing continuous function called $\mathcal{K}^4$-Devil's staircase function and defined as:

Let $\mathcal{J}_0^4(x) = x$ , $x \in \mathcal{K}_1^4 = [0,1]$and for each $v \in N$:

$$\mathcal{J}_v^4(x) = \begin{cases} \dfrac{4^v x + g}{2^v} & x \in \mathcal{K}_\ell^4 \\ \dfrac{2c+1}{2^b} & x \in \mathcal{Z}_{\ell^*}^4 \end{cases},$$

where, $g = c - 3e$ , $\ell^* = c + 2^{b-1}$, $0 \le c < 2^{b-1}$, and $1 \le b \le v$.

For example, let $v = 1$: $\{c: 0 \le c < 2^v\} = \{0,1\}$, then: $c = 0 \equiv 0.2^0, e = \sum_{t=0}^{\infty} 4^t e_t = 0.4^0 = 0$ and when $c = 1 \equiv 1.2^0, e = \sum_{t=0}^{\infty} 4^t e_t = 1.4^0 = 1$, we have

$$\mathcal{J}_1^4(x) = \begin{cases} 2x & x \in \mathcal{K}_2^4 \\ 2x - 1 & x \in \mathcal{K}_3^4. \\ 2^{-1} & x \in \mathcal{Z}_1^4 \end{cases}$$

# 4 $\mathcal{N}^4$-FOURIER SERIES

In this section, we will introduce an important kind of orthogonal system called $\mathcal{N}^4$-system. The representation and some properties for this system with their proofs are given. Finally, defining a $\mathcal{N}^4$-Fourier transform for an absolutely integrable function on [0,1] concerning to the measure of $\mathcal{K}^4$-Devil's staircase function on R.C.S.

For each $v \in \mathrm{N}_0$, $0 \le c < 2^v$, we write the dyadic expansion of $c$:

$$c = \sum_{t=0}^{\gamma} 2^v c_t,$$

with its Gray code

$$w_c^G = \{c_\gamma \oplus c_{\gamma+1}, c_{\gamma-1} \oplus c_\gamma, \cdots, c_1 \oplus c_2, c_0 \oplus c_1\} = \{c_\gamma^*, c_{\gamma-1}^*, \cdots, c_1^*, c_0^*\}, c_{\gamma+1} = 0.$$

Then, we obtain $\mathcal{P}^4 = \{\mathcal{P}_v^4(x)\}_{v=0}^{\infty}$-system:

$$\mathcal{P}_0^4(x) \equiv 1, \forall\, x \in [0,1]$$

$$\mathcal{P}_v^4(x) = \begin{cases} (-1)^{\sum_{t=0}^{v} c_t^*} & x \in \mathcal{K}_\ell^4 \\ 0 & o.w. \end{cases}$$

Lemma 4.1:

1) The integral of $\mathcal{P}^4$-system is zero concerning to the measure of $\mathcal{K}^4$-Devil's staircase function on R.C.S. i.e.:

$$\int_0^1 \mathcal{P}_v^4(x)\, d\mu = 0, \forall\, v: v \in N.$$

2) Each $\mathcal{P}^4$-system takes on the value {+1,-1} except at the jumps, where it takes the value 0.
3) $\mathcal{P}_v^4(0) = 1, \forall\, v: v \in N_0$.

Theorem 4.1: $\mathcal{P}^4$-system is orthonormal basis on [0,1] with respect to measure of $\mathcal{K}^4$-Devil's staircase function on R.C.S.

Proof: If $\mathbb{i} \ne \mathbb{j}$, $(\mathbb{j} > \mathbb{i})$, $\{c_{\mathbb{i}}: 0 \le c_{\mathbb{i}} < 2^{v_{\mathbb{i}}}\}, \{c_{\mathbb{j}}: 0 \le c_{\mathbb{j}} < 2^{v_{\mathbb{j}}}\} \subseteq \{c: 0 \le c < 2^v\}$ and

$$\mathcal{K}_{\ell_{\mathbb{i}}}^4 = [\tfrac{3e_{\mathbb{i}}}{4^v}, \tfrac{3e_{\mathbb{i}}+1}{4^v}],\ 0 \le c_{\mathbb{i}} < 2^{v_{\mathbb{i}}},$$

$$\mathcal{K}_{\ell_{\mathbb{j}}}^4 = [\tfrac{3e_{\mathbb{j}}}{4^v}, \tfrac{3e_{\mathbb{j}}+1}{4^v}],\ 0 \le c_{\mathbb{j}} < 2^{v_{\mathbb{j}}},$$

where, $\ell_{\mathbb{i}} = 2^{v_{\mathbb{i}}} + c_{\mathbb{i}}$, and $\ell_{\mathbb{j}} = 2^{v_{\mathbb{j}}} + c_{\mathbb{j}}$, then we have the following cases:

Table 1: 4-adic-type Cantor-like sets.

| $\mathscr{v}$ | | $\{c: 0 \leq$ $< 2^{\mathscr{v}}\}$ | $c = \sum_{t=0}^{\infty} 2^t c_t$ | $e = \sum_{t=0}^{\infty} 4^t e_t$ | $\mathcal{K}_{\ell}^4$ | $\mathcal{Z}_{\ell}^4$ |
|---|---|---|---|---|---|---|
| 0 | | 0 | $c = 0.2^0$ | $e = 0.4^0 = 0$ | $\mathcal{K}_1^4 = [0,1]$ | |
| 1 | | 0 | $c = 0.2^0 = 0$ | $e = 0.4^0 = 0$ | $\mathcal{K}_2^4 = [0,1/4]$ | $\mathcal{Z}_1^4 = (1/4,3/4)$ |
| | | 1 | $c = 1.2^0 = 1$ | $e = 1.4^0 = 1$ | $\mathcal{K}_3^4 = [3/4,1]$ | |
| 2 | | 0 | $c = 0.2^0 = 0$ | $e = 0.4^0 = 0$ | $\mathcal{K}_4^4 = [0,1/4^2]$ | $\mathcal{Z}_2^4 = (1/4^2,3/4^2)$ |
| | | 1 | $c = 1.2^0 = 1$ | $e = 1.4^0 = 1$ | $\mathcal{K}_5^4 = [3/4^2,4/4^2]$ | $\mathcal{Z}_3^4 = (13/4^2,15/4^2)$ |
| | | 2 | $c = 0.2^0 + 1.2^1 = 2$ | $e = 0.4^0 + 1.4^1 = 4$ | $\mathcal{K}_6^4 = [12/4^2,13/4^2]$ | |
| | | 3 | $c = 1.2^0 + 1.2^1 = 3$ | $e = 1.4^0 + 1.4^1 = 5$ | $\mathcal{K}_7^4 = [15/4^2,1]$ | |
| 3 | | 0 | $c = 0.2^0 = 0$ | $e = 0.4^0 = 0$ | $\mathcal{K}_8^4 = [0,1/4^3]$ | $\mathcal{Z}_4^4 = (1/4^3,3/4^3)$ |
| | | 1 | $c = 1.2^0 = 1$ | $e = 1.4^0 = 1$ | $\mathcal{K}_9^4 = [3/4^3,4/4^3]$ | $\mathcal{Z}_5^4 = (13/4^3,15/4^3)$ |
| | | 2 | $c = 0.2^0 + 1.2^1 = 2$ | $e = 0.4^0 + 1.4^1 = 4$ | $\mathcal{K}_{10}^4 = [12/4^3,13/4^3]$ | $\mathcal{Z}_6^4 = (49/4^3,51/4^3)$ |
| | | 3 | $c = 1.2^0 + 1.2^1 = 3$ | $e = 1.4^0 + 1.4^1 = 5$ | $\mathcal{K}_{11}^4 = [15/4^3,16/4^3]$ | $\mathcal{Z}_7^4 = (61/4^3,63/4^3)$ |
| | | 4 | $c = 0.2^0 + 0.2^1 + 0.2^2 = 4$ | $e = 0.4^0 + 0.4^1 + 1.4^2 = 16$ | $\mathcal{K}_{12}^4 = [48/4^3,49/4^3]$ | |
| | | 5 | $c = 1.2^0 + 0.2^1 + 1.2^2 = 5$ | $e = 1.4^0 + 0.4^1 + 1.4^2 = 17$ | $\mathcal{K}_{13}^4 = [51/4^3,52/4^3]$ | |
| | | 6 | $c = 0.2^0 + 1.2^1 + 1.2^2 = 6$ | $e = 0.4^0 + 1.4^1 + 1.4^2 = 20$ | $\mathcal{K}_{14}^4 = [60/4^3,61/4^3]$ | |
| | | 7 | $c = 1.2^0 + 1.2^1 + 1.2^2 = 7$ | $e = 1.4^0 + 1.4^1 + 1.4^2 = 21$ | $\mathcal{K}_{15}^4 = [63/4^3,1]$ | |

Case 1: If $\exists \mathscr{v}_{\mathbb{i}}, \mathscr{v}_{\mathbb{j}}, e_{\mathbb{i}},$ and $e_{\mathbb{j}}$ satisfy $\frac{3e_{\mathbb{i}}}{4^{\mathscr{v}}} = \frac{3e_{\mathbb{j}}}{4^{\mathscr{v}}}$ Then, we get $\mathcal{K}_{\ell_{\mathbb{j}}}^4 \subseteq \mathcal{K}_{\ell_{\mathbb{i}}}^4$ with $\mathcal{P}_{\mathbb{j}}^4(x) = 1$ and $\mathcal{P}_{\mathbb{i}}^4(x)$ is non-zero along the interval $\mathcal{K}_{\ell_{\mathbb{i}}}^4$.

Case 2: If $\exists \mathscr{v}_{\mathbb{i}}, \mathscr{v}_{\mathbb{j}}, e_{\mathbb{i}},$ and $e_{\mathbb{j}}$ satisfy the following relation $\frac{3e_{\mathbb{i}}+1}{4^{\mathscr{v}}} = \frac{3e_{\mathbb{j}}+1}{4^{\mathscr{v}}}$ Then, we obtain

$$\mathcal{K}_{\ell_{\mathbb{j}}}^4 \subseteq \mathcal{K}_{\ell_{\mathbb{i}}}^4 ,$$

with $\mathcal{P}_{\mathbb{j}}^4(x) = -1$, and $\mathcal{P}_{\mathbb{i}}^4(x)$ is non-zero along the interval $\mathcal{K}_{\ell_{\mathbb{i}}}^4$. Therefore, from case 1 and case 2, we have

$$\int_0^1 \mathcal{P}_{\mathbb{i}}^4(x)\mathcal{P}_{\mathbb{j}}^4(x)\, d\mu = \int_{\mathcal{K}_{\ell}^4} \mathcal{P}_{\mathbb{i}}^4(x)\mathcal{P}_{\mathbb{j}}^4(x)\, d\mu = \mp \int_{\mathcal{K}_{\ell_{\mathbb{i}}}^4} \mathcal{P}_{\mathbb{i}}^4(x)\, d\mu.$$

By using Lemma 4.1 case 1, we get

$$\int_0^1 \mathcal{P}_{\mathbb{i}}^4(x)\mathcal{P}_{\mathbb{j}}^4(x)\, d\mu = 0, \ \forall \mathbb{i} \neq \mathbb{j} .$$

If $\mathbb{i} = \mathbb{j}$, then put $\mathcal{T} = \{\mathcal{K}_{\propto}^4, \propto = 2^{\mathscr{v}+1} + c, 0 \leq c < 2^{\mathscr{v}+1}\}$, and

$$\int_0^1 \mathcal{P}_{\mathbb{i}}^4(x)\mathcal{P}_{\mathbb{j}}^4(x)\, d\mu = \int_0^1 (\mathcal{P}_{\mathbb{i}}^4(x))^2 d\mu = \sum_{\mathcal{B}\in\mathcal{T}} \int_{\mathcal{B}} d\mu = \sum_{\mathcal{B}\in\mathcal{T}} \mu(\mathcal{B}) = 1 \ \blacksquare.$$

Lemma 4.2: $\mathcal{P}^4$-system is linearly independent on [0,1].

We shall find that $\mathcal{P}^4$-system forms a sub set of $\mathcal{N}^4$-system which is complete orthonormal system on [0,1]. Therefor the $\mathcal{P}^4$-system is in complete orthonormal on [0,1].

Remark: Paley [18], defined a new method to generate Walsh functions. His definition is based on the finite product of Rademacher functions. The Rademacher functions were described by the German mathematician H. Rademacher [18-19], in which he defined a system of orthogonal functions, each function taking only the values +1 or -1, except at jumps, where they take on the value zero.

By using Paley sense, define $\mathcal{N}^4$-system:

$$\mathcal{P}_0^4(x) = \mathcal{N}_0^4(x) \equiv 1, \forall\, x \in [0,1].$$

For $\mathscr{v} > 0$, write the dyadic expansion of $\mathscr{v}$:

$$\mathscr{v} = \sum_{t=0}^{t_{\mathscr{v}}} 2^t \mathscr{v}_t .$$

Where, $t_{\mathscr{v}} = [log_2 \mathscr{v}]$, $\mathscr{v}_t \in \mathbb{G}$, and

$$\mathcal{N}_{\mathscr{v}}^4(x) = \mathcal{P}_{t_{\mathscr{v}}+1}^4(x) \prod_{t=0}^{t_{\mathscr{v}}-1} (\mathcal{P}_{t+1}^4(x))^{\mathscr{v}_t}. \qquad (1)$$

Equation (1) has the advantage that, since $\mathcal{P}^4$-system is regular and periodic, it is easy to remember them and their products are easy form.

Lemma 4.3:

i) $\mathcal{N}_{\mathscr{v}}^4(x) = [\mathcal{H}_{\hbar}]_{\mathscr{v}z} = \frac{1}{\sqrt{\hbar}} cas\left(\frac{2\mathscr{v}z}{\hbar}\right), 0 \le \mathscr{v}, z < \hbar - 1,$

where, $\hbar$- is the order of Hartley transform of type $\mathcal{H}^I$ [20] and $cas(x) = \sin(x) + \cos(x)$, for each $x = \frac{3e}{4^{\mathscr{v}}}$ , $\mathscr{v} \in N_0$, where, $e = \sum_{t=0}^{\infty} 4^t e_t$, with $e_t$ being the dyadic coefficients of $c$ in binary system:

$$c = \sum_{t=0}^{\infty} 2^t c_t \text{ , } c_t \in \mathbb{G}.$$

ii) $\mathcal{N}_{\mathscr{v}}^4(0) = 1$, for all $\mathscr{v}: \mathscr{v} \in N_0$.

iii) $\mathcal{N}^4 = \{\mathcal{N}_{\mathscr{v}}^4(x)\}_{\mathscr{v}=0}^{\infty}$-system forms a group with respect to multiplication.

iiii) $\int_0^1 \mathcal{N}_{\mathscr{v}}^4(x) d\mu = 0.$

Proof: one can simply show that i), ii), and iiii) from definition of Hartley transform of type $\mathcal{H}^I$, relation (1), and in relation (1) respectively.

iii) For each $i, j \ge 0$, we write the binary number of $i, j$ as: $(i)_b = \{\lambda_{\xi_i}, \cdots, \lambda_0\}$ and $(j)_b = \{\delta_{\theta_j}, \cdots, \delta_0\}$, where, the digits $\lambda_{\xi_i}, \cdots, \lambda_0$ , $\delta_{\theta_j}, \cdots, \delta_0$ are in $\mathbb{G}$. We define the multiplication as:

$$\mathcal{N}_{(i)_b}^4(x).\mathcal{N}_{(j)_b}^4(x) = \mathcal{N}_{(i)_b \oplus (j)_b}^4(x) = \mathcal{N}_{(j)_b}^4(x).\mathcal{N}_{(i)_b}^4(x), \quad \forall\, x \in \mathcal{K}_1^4.$$

$\mathcal{N}_0^4(x)$ is the identity element of $\mathcal{N}^4$-system:

$$\mathcal{N}_{(0)_b}^4(x).\mathcal{N}_{(j)_b}^4(x) = \mathcal{N}_{(0)_b \oplus (j)_b}^4(x) = \mathcal{N}_{(j)_b}^4(x)$$

For each $\mathcal{N}_{(j)_b}^4(x) \in \mathcal{N}^4$-system.

The inverse element of $\mathcal{N}_j^4(x)$ is $\mathcal{N}_j^4(x)$ it self:

$$\mathcal{N}_{(j)_b}^4(x).\mathcal{N}_{(j)_b}^4(x) = \mathcal{N}_{(j)_b \oplus (j)_b}^4(x) = \mathcal{N}_{(0)_b}^4(x) = \mathcal{N}_0^4(x) = 1.$$

is associative: For each $i, j, p \ge 0$, we have

$$(\mathcal{N}_{(i)_b}^4(x).\mathcal{N}_{(j)_b}^4(x)).\mathcal{N}_{(p)_b}^4(x) = (\mathcal{N}_{(i)_b \oplus (j)_b}^4(x)) .\mathcal{N}_{(p)_b}^4(x) = \mathcal{N}_{(i)_b \oplus (j)_b \oplus (p)_b}^4(x) = (\mathcal{N}_{(i)_b}^4(x)).\mathcal{N}_{(j)_b \oplus (p)_b}^4 = \mathcal{N}_{(i)_b}^4(x)(\mathcal{N}_{(j)_b}^4(x).\mathcal{N}_{(p)_b}^4(x)).$$

Therefore $(\mathcal{N}^4, .)$ is a group which is abelian as it a obvious from the definition of associative.

Theorem 4.2: $\mathcal{N}^4$-system is orthonormal basis on [0,1] concerning to measure of $\mathcal{K}^4$-Devil's staircase function on R.C.S.

Proof: For each $i, j \ge 0$, we write the binary number of $i, j$ as:
$(i)_b = \{\lambda_{\xi_i}, \cdots, \lambda_0\}$ and $(j)_b = \{\delta_{\theta_j}, \cdots, \delta_0\}$, where, the digits $\lambda_{\xi_i}, \cdots, \lambda_0$ , $\delta_{\theta_j}, \cdots, \delta_0$ are in $\mathbb{G}$ and via lemma 4.3 iii), we obtain:
If $i = j$, then, we have

$$\int_0^1 \mathcal{N}_i^4(x)\, \mathcal{N}_j^4(x)\, d\mu = \int_0^1 \mathcal{N}_{(i)_b}^4(x)\, \mathcal{N}_{(j)_b}^4\, d\mu = \int_0^1 \mathcal{N}_{(i)_b \oplus (j)_b}^4(x)\, d\mu = \int_0^1 \mathcal{N}_{(i)_b \oplus (i)_b}^4(x)\, d\mu = \int_0^1 \mathcal{N}_{(0)_b}^4(x) d\mu = \int_0^1 d\mu = \mathcal{J}_0^4(1) - \mathcal{J}_0^4(0) = 1$$

If $i \ne j$, then

$$\int_0^1 \mathcal{N}_i^4(x)\, \mathcal{N}_j^4(x)\, d\mu = \int_0^1 \mathcal{N}_{(i)_b}^4(x)\, \mathcal{N}_{(j)_b}^4\, d\mu = \int_0^1 \mathcal{N}_{(i)_b \oplus (j)_b}^4(x)\, d\mu$$

From section 2, there exist a unique natural number $\zeta$ such that :

$\zeta \equiv (\zeta)_b = (i)_b \oplus (j)_b$ , therefore

$$\int_0^1 \mathcal{N}_i^4(x)\, \mathcal{N}_j^4(x)\, d\mu = \int_0^1 \mathcal{N}_{(i)_b \oplus (j)_b}^4(x)\, d\mu = \int_0^1 \mathcal{N}_{(\zeta)_b}^4(x)\, d\mu = \int_0^1 \mathcal{N}_\zeta^4(x)\, d\mu.$$

Via using lemma 4.3 iiii), we obtain

$$\int_0^1 \mathcal{N}_i^4(x)\, \mathcal{N}_j^4(x)\, d\mu = 0, \forall i \ne j \;\blacksquare.$$

Theorem 4.3: $\mathcal{N}^4$-system is a complete orthonormal system concerning to measure on a 4-adic-type Cantor like set $\mathcal{K}^4$.

Proof: Assume that $\psi$ is the integrable function, and

$$\int_0^1 \psi(x)\mathcal{N}_i^4(x) d\mu = 0,\ 0 \le i < 2^{\mathscr{v}}. \qquad (2)$$

Moreover, assume

$$\mathfrak{N}(x) = \int_0^x \psi(t) d\mu, \quad \forall\, x \in \mathcal{K}_\ell^4.$$

Therefore $\mathfrak{N}(x)$ have two properties [22]

i) $\mathfrak{N}(x)$ is continuous on [0,1];

ii) $\mathfrak{N}(x)$ is differentiable at every $x \in [0,1]$ at which $\psi(x)$ is continuous and $\mathfrak{N}'(x) = \psi(x)$.

It then follows from our assumptions that

$$\int_0^1 \psi(x)\mathcal{N}_i^4(x) d\mu = \sum_{\vartheta=0}^{2^{\mathscr{v}}-1} \int_{\mathcal{K}_{r^*}^4} \psi(x)\mathcal{N}_i^4\left(\frac{3e}{4^{\mathscr{v}}}\right) d\mu\,, \quad 0 \le i < 2^{\mathscr{v}}\,, r^* = 2^{\mathscr{v}} + \vartheta$$

$$= \sum_{\vartheta=0}^{2^{\mho}-1} \int_{\frac{3e}{4^{\mho}}}^{\frac{3e+1}{4^{\mho}}} \psi(x)\mathcal{N}_i^4\left(\frac{3e}{4^{\mho}}\right) d\mu = 0.$$

From the property of linearly independent and lemma 4.3 i), we get

$$\mathcal{N}_i^4\left(\frac{3e}{4^{\mho}}\right) = [\mathcal{H}_{\hbar}]_{\mho z} \neq 0\ ,\ 0 \leq i < 2^{\mho}.$$

Also, the numbers $[\mathcal{H}_{\hbar}]_{\mho z}$ are linearly independent. Therefore

$$\int_{\frac{3e}{4^{\mho}}}^{\frac{3e+1}{4^{\mho}}} \psi(x)\mathcal{N}_i^4\left(\frac{3e}{4^{\mho}}\right) d\mu = 0. \qquad (3)$$

Assume that, the relation in (2). Using (3), we have

$$\mathfrak{N}\left(\frac{3e}{4^{\mho}}\right) = \mathfrak{N}\left(\frac{3e+1}{4^{\mho}}\right).$$

From which Alexits [23] concludes that $\mathfrak{N}(x) = 0$, and $\mathfrak{N}'(x) = 0$, $\forall x \in [0,1]$. Since $\mathfrak{N}'(x) = \psi(x)$ at every $x \in [0,1]$ in which $\psi(x)$ is continuous implies that $\psi(x) = 0$, at which $\psi(x)$ is continuous, then $\psi = \mathbf{0}$ in $\mathcal{K}_{\ell}^4$ ■.

Any absolutely integrable function $\mathcal{F}$ on the interval [0,1] has a Fourier series in $\mathcal{N}^4$-system:

$$\mathcal{F}(x) \sim \sum_{i=0}^{\infty} \omega_i\, \mathcal{N}_i^4(x). \qquad (4)$$

where,

$$\omega_i = \int_0^1 \mathcal{F}(x)\mathcal{N}_i^4(x) d\mu\ .$$

Are the $\mathcal{N}^4$-Fourier coefficients of $\mathcal{F}(x)$ and the series in relation (4) is called $\mathcal{N}^4$-Fourier series of $\mathcal{F}(x)$.

The $\rho$-th partial sum of the $\mathcal{N}^4$-Fourier series of a function $\mathcal{F}(x)$ will be represented via:

$$\Gamma_\rho = \sum_{i=0}^{\rho} \omega_i\, \mathcal{N}_i^4(x). \qquad (5)$$

# 5 ORTHOGONAL MULTIPLEXING SYSTEM (O.M.S.) IN $\mathcal{N}^4$-FOURIER SERIES

Since a $\mathcal{N}^4$-system forms complete orthonormal functions on R.C.S. takes only the values +1 and -1 which are likely to be well suited to multiplexing systems as well as the numerical $\mathcal{N}^4$-Fourier transmission and numerical sequence shifting of signals require summations and subtraction only.

The input signals $\mathcal{F}_i$ , $i = 0,1,\cdots,k-1, k^z, z$ is a positive integer, passes first through filter. Then the set of functions $\mathcal{N}_i^4(x)$ are multiplied via $\mathcal{F}_i$ via the multipliers $\mathcal{M}$. The product is $\mathcal{F}_i\mathcal{N}_i^4(x)$ are summed and transmitted to obtain a signal $\mathcal{F}(x)$:

$$\mathcal{F}(x) = \sum_{i=0}^{k-1} \mathcal{F}_i\, \mathcal{N}_i^4(x). \qquad (6)$$

In order to recover $\mathcal{F}_i$, the signal $\mathcal{F}(x)$ is multiplied by $\mathcal{N}_j^4(x)$, where, $j = 0,1,\cdots,k-1$, and integrating the product in the normalized interval $0 \leq x \leq 1$:

$$\int_0^1 \mathcal{F}(x)\mathcal{N}_i^4(x) d\mu = \int_0^1 \sum_{i=0}^{k-1} \mathcal{F}_i\, \mathcal{N}_i^4(x)\mathcal{N}_j^4(x) d\mu.$$

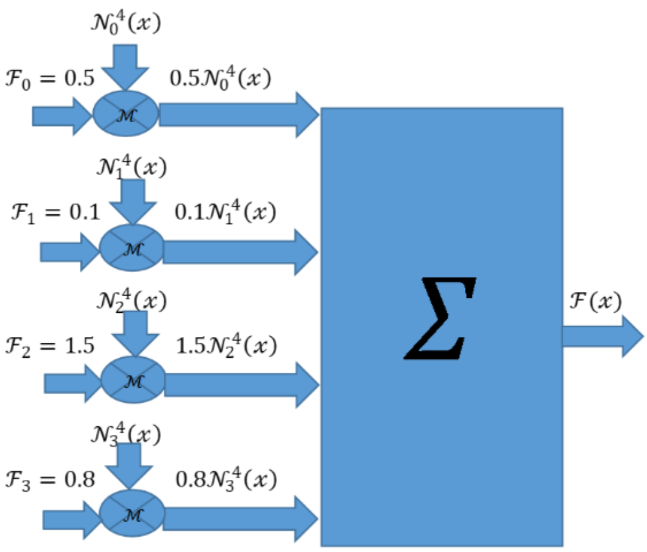


Figure 1: $\mathcal{N}^4$- carrier O.M.S.

By using theorem (4.2), the output of each de-multiplexing channel $i = j$, will be:

$$\int_0^1 \mathcal{F}(x)\mathcal{N}_i^4(x)d\mu = \mathcal{F}_i \ .$$

Example 1: consider the input signals of $\mathcal{F}(x)$:

$$\mathcal{F}_0 = 0.5, \mathcal{F}_1 = 0.1, \mathcal{F}_2 = 1.5, \mathcal{F}_3 = 0.8.$$

Each signal is multiplied via $4 - \mathcal{N}^4$-functions of order 4: the situation is depicted in Figure 1, and the waveform details for each channel are shown in Table 2. The total signal $\mathcal{F}(x)$ in Table 2 and Figure 1 denotes the sum of all four active channel signals, and designed in Figure 2.

Table 2: Detail of $\mathcal{N}^4$- carrier O.M.S. of Figure 1.

| Time $\rightarrow$ | $x = 0$ | $x = 1/4$ | $x = 3/4$ | $x = 1$ |
|---|---|---|---|---|
| $0.5\mathcal{N}_0^4(x)$ | 0.5 | 0.5 | 0.5 | 0.5 |
| $0.1\mathcal{N}_1^4(x)$ | 0.1 | 0.1 | -0.1 | -0.1 |
| $1.5\mathcal{N}_2^4(x)$ | 1.5 | -1.5 | 1.5 | -1.5 |
| $0.8\mathcal{N}_3^4(x)$ | 0.8 | -0.8 | -0.8 | 0.8 |
| $\mathcal{F}(x)$ | 2.9 | -1.7 | 1.1 | -0.3 |

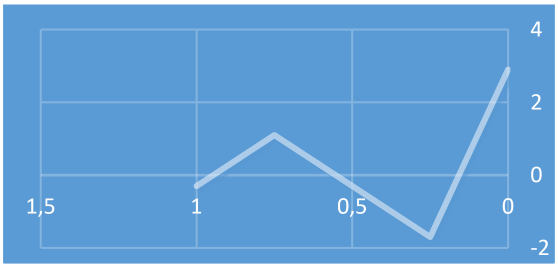


Figure 2: Total multiplexing signal of example 1.

## 6 CONCLUSIONS

This paper introduces a novel construction of piecewise constant orthonormal functions based on the rescaling Cantor set (R.C.S.), particularly utilizing 4-adic-type Cantor-like sets. The developed orthogonal system demonstrates strong properties of completeness, orthonormality, and linear independence with respect to the measure defined by the –Devil's staircase function. These foundational properties affirm the mathematical robustness and applicability of the constructed system. The paper presents the results of research on Rescaling Cantor set (R.C.S) of 4-adic-type Cantor-like set type. We obtained characteristic and in some cases criteria of completeness, orthonormality and basis property of $\mathcal{N}^4$-system concerning to measure of $\mathcal{K}^4$-Devil's staircase function on R.C.S. Many independent signals over communication channel in Multiplexing system are carried based on regularization of functions generating this system. Compared with the method of Fourier transform [18], the suggested approach is simpler in theory and easier in implementation. It is believed that this is the first time in using the $\mathcal{N}^4$- series to approach the most interesting problem in O.M.S. In addition to all the possibilities discussed above, we shall also investigate more properties of the $\mathcal{N}^4$-system and try to form a construction for the $\mathscr{k}$- adic- type Cantor-like set. Also, we may try to extend our work to include $\mathcal{N}^n$-system, where, $n \geq 4$. Compared with conventional Fourier-based methods, the –system offers theoretical simplicity, faster computational performance, and the potential for scalable extension to other adic-type Cantor sets. Moreover, this approach opens avenues for future research in signal processing, coding theory, and nonlinear dynamic analysis over fractal domains.

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