A New Approach with Software Implementation to Extend the Pythagorean Theorem in Multi-Dimensions

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Abstract: The Pythagorean theorem, which states that "for any right triangle, the square of the hypotenuse is equal to the sum of the squares of the lengths of the two legs," applies only within a single plane (i.e., in two dimensions). This paper proposes an extension of the Pythagorean theorem to multiple planes and dimensions by introducing a novel concept called Abd's 1st Intuition or Abd's First Theory (AFT). In this extended version, a finite sequence of right triangles is constructed, starting from a specific initial right triangle with two legs and a hypotenuse. Each subsequent right triangle is formed with one leg as the hypotenuse of the previous triangle and the other leg as a new value. This process continues such that the n-th right triangle has legs corresponding to the previous hypotenuse and a new dimension's value, resulting in a new hypotenuse. Each transition to a new plane introduces a new dimension, culminating in an extended form of the theorem: the square of the hypotenuse of this final right triangle is equal to the sum of the squares of the legs from all the constructed triangles. Additionally, a sub-exponential growth pattern for the sequence of hypotenuses is derived, demonstrating that their growth rate depends on the acute angle adjacent to the legs. For the special case of isosceles right triangles, the hypotenuse of the n-th triangle can be explicitly calculated from the first leg of the base triangle. This generalized theorem is intended for applications in wireless and wired communication networks, cryptography, communications security, and other areas within applied programming and computer science.

1 INTRODUCTION

The tremendous development in all fields during the past four decades as a result of the widespread and rapid spread of information and communications technology and wireless communication networks required a reconsideration of mathematical theories and theorems in a way that serves the other applying sciences in our current era [1]. The importance of the Pythagorean theorem (PT) lies in its mathematical and engineering applications, such as calculating right triangles, trigonometric ratios, and finding the distances between points and determine angles and lengths in the engineering drawing [2] Also, this theorem is used in the design of buildings, where it is used by architects and construction engineers in the civil engineering, also in the electrical engineering to design the coils of the generators and motors [3],[4]. It is also used in the space engineering for calculating the distances among the celestial bodies [5], the movement of spacecraft, fighter jets, autopilot system, projectiles, missiles, submarines, and navigation at the sea [6], [7]. In addition, this theory appears in the scanning technology and wireless networks to determine the right and accurate distance between the source and the receiver with the necessary corrections for that [8], [9], while this theorem represents the basis for all Global Positioning System (GPS) measurements [10].

In this research, we present a new mathematical perspective on the PT by extending it to a finite number of planes and dimensions. We aim to make the theorem more comprehensive and then prove our new conjecture using the perfect square formula and mathematical induction, supported by illustrative examples. Subsequently, we implement the theorem in Python to generalize its application and explore its potential uses in various fields Such as architecture, space engineering, navigation, rocket engineering, wireless communication networks, artificial intelligence, and others.

2 PYTHAGOREAN THEOREM AND ITS VARIOUS EXTANSION

The Pythagorean Theorem (PT) is one of the most important mathematical theorems developed in the history of mathematics. This theorem is named after Pythagoras, a Greek mathematician from the 6th century BC who developed it in the 5th century BC (Nanda 2016). The PT of any right triangle as given in Figure 1 (which is applied in one plane), states that the square of the hypotenuse, h_1 (the side opposite the right angle) is equal to the sum of squares of the other two sides, x_1 and x_2 (we call it legs) according to the mathematical expression [11, [12];

$$x_1^2 + x_2^2 = h_1^2 \tag{1}$$

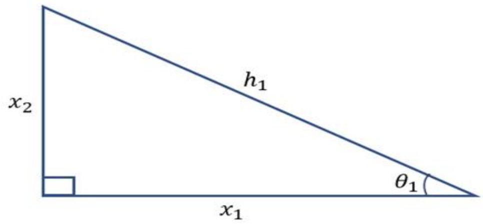

Figure 1: The right triangle.

It should be noted that the inverse of the PT is also true, meaning that a triangle is right-angled if the square of one of its sides (the longest side) is equal to the sum of the squares of the other two sides. While the square of the sum of two terms is equal to the sum of both its squares plus twice multiply them, as shown in the mathematical formula below [13];

$$(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 \tag{2}$$

This theorem has been extended and generalized in various ways over the centuries, leading to a rich field of mathematical research. The PT can be generalized to higher dimensions. In an n-dimensional Euclidean space, the distance between two points can be calculated using the generalized Pythagorean Theorem, also known as the Euclidean distance formula [14]. The Law of Cosines is a generalization of the Pythagorean Theorem to non-right triangles [15]. In non-Euclidean geometries, such as spherical and hyperbolic geometry, the PT does not hold in its classical form. However, there are analogous results in spherical geometry and in hyperbolic geometry that are critical in the study of curved spaces, such as in general relativity [16]. In number theory, there is Pythagorean triples which have been studied extensively and another extension of this concept is including the study of primitive Pythagorean triples and their properties in modular arithmetic and algebraic number theory [17].

In the concept of vector space and complex numbers, we can extend the PT in more than one level and in several dimensions via utilizing the idea of inner product associated with the Cauchy-Schwartz inequality and the parallelogram rule [18]. Also, we can extend the PT in the algebraic geometry to include various cases, like the Pythagorean technicals defined by equations similar to the Pythagorean conformity. Therefore, this scope shows the geometric characteristics relationship with algebraic texture [19]. Extensions of the PT are vastly utilized in physical fields, especially with vector amounts like the forces with their moments, the velocities with their accelerations, and electromagnetic fields with their directional effects. For example, Minkowski space (in special relativity) represents modulated version of the PT to calculate space-time periods [20]. Therefore, the PT represents the backbone of the math with the geometry as all, where its extensions into multi-dimensions via various planes, non-Euclidean spaces, algebraic geometry, number theory, and physical fields explain its deep effect and the wide scope of its applications. The continued studies of all these deepen our understanding of geometry and also join it to other mathematical corrections and feasible applications. In this work, we have expanded the PT into many planes via adding more right triangles that share sides with each other.

3 NEW EXTANDED VERSION OF THE PYTHAGOREAN THEOREM: ABD'S 1ST INTUITION

Our intuition gives attractive extension of the PT via the addition of finitely verity planes, each including a right triangle. This extension generates a multi-dimensional build that expands the traditional 2-dimension version of the PT.

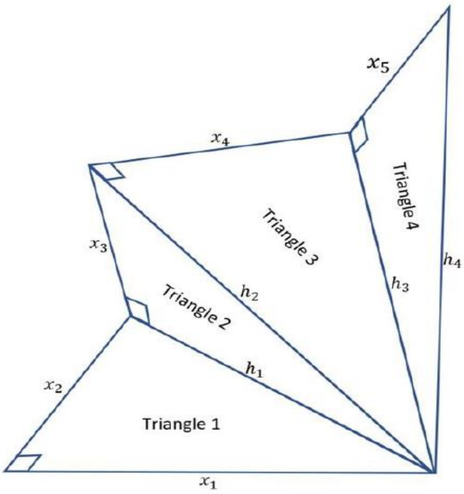

Figure 2: Creating four right triangles to illustrate Abd's 1st intuition.

Figure 2 illustrates our idea in the expanding mechanism, beginning with the foundation right triangle in the xy-plane (the first plane) with two right legs x_1 and x_2 and hypotenuse h_1, we build a second plane including a second right triangle with two right legs x_3 and h_1 and hypotenuse h_2. This second right triangle shares the hypotenuse h_1 with the base triangle, establishing a connection between the two. Subsequently, a third plane is introduced with a right triangle having legs x_4 and h_2, and hypotenuse h_3. This process continues, with each new triangle connecting to the previous one via a shared hypotenuse. Ultimately, this construction results in a geometric structure consisting of finitely many right triangles (say k triangles), each connected through shared hypotenuses. Refer to Figure 2 for a visual representation of this geometric construction of 4 right triangles.

In our institution, we conjecture that the sum of the squares of x_i (for $i = 1,2,\cdots,k$) is equal to the square of h_k, as given as follows

$$x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = h_k^2 \quad (3)$$

This suggests that the extension not only creates a multi-dimensional structure but also preserves a relationship akin to the original Pythagorean Theorem, where the cumulative effect of the legs in each plane equates to the hypotenuse in the final plane.

3.1 Adopting the Perfect Square Formula

By using perfect squares law, we may leverage the familiar properties of squares to understand our institution, the extended PT in a higher-dimensional context. We may see that as we add more planes and triangles, the relationship $\Sigma_{i=1}^{k+1} x_i^2 = h_k^2$ remains consistent, showing that the geometric extension preserves the Pythagorean relationship in a more complex structure.

For the first right triangle, we have $x_1 = h_1 \cos\theta_1$ and $x_2 = h_1 \sin\theta_1$. From the perfect square law, we know that

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 \quad (4)$$

Therefore we have

$$x_1^2 + x_2^2 = (h_1\cos\theta_1 + h_1\sin\theta_1)^2 - 2(h_1\cos\theta_1)(h_1\sin\theta_1)$$

This shows that

$$x_1^2 + x_2^2 = h_1^2 \quad (5)$$

For the second right triangle, we have $h_1 = h_2\cos\theta_2$ and $x_3 = h_2\sin\theta_2$. From the perfect square law, we know that

$$h_1^2 + x_3^2 = (h_1 + x_3)^2 - 2h_1x_3 \quad (6)$$

Therefore we have

$$h_1^2 + x_3^2 = (h_2\cos\theta_2 + h_2\sin\theta_2)^2 - 2(h_2\cos\theta_2)(h_2\sin\theta_2) \quad (7)$$

This shows that

$$h_1^2 + x_3^2 = h_2^2 \quad (8)$$

Then substitute (5) into the (8) yields

$$x_1^2 + x_2^2 + x_3^2 = h_2^2 \quad (9)$$

Moving to the next triangles, say to the k-th, we may substitute the finding in the $(k-1)$-th triangle to show that

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_{k+1}^2 = h_k^2 \quad (10)$$

This Equation represents our intuition, which is that "The square of the length of the hypotenuse in a right triangle resulting from several right triangles in space equals the sum of the squares of the lengths for all its legs (right sides)", hence k represents the number of right triangles or the number of levels in space and $k+1$ represents the number of dimensions, while x indicates the number of legs of the right triangles, and h represents the hypotenuse of the last triangle. Thus, the Intuition becomes as follows

$$x_1^2 + \sum_{i=1}^{i=k} x = h_k^2. \quad (11)$$

To illustrate the mechanism of this expansion and its stages, we present an example for that:

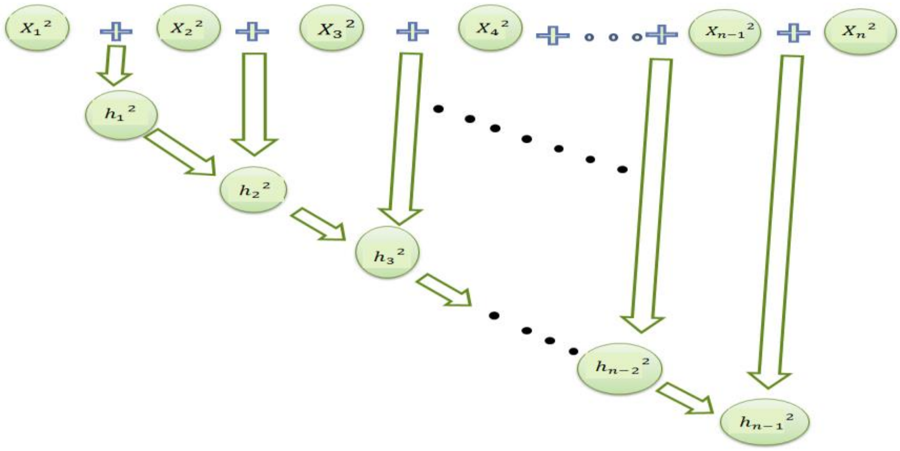

Figure 3: The action mechanism of Abd's 1st intuition.

Let, $k = 3$, $\mathrm{x}_1 = 4$, and $\mathrm{x}_2 = 3$.

$$h_1{}^2 = X_1{}^2 + X_2{}^2 \leftrightarrow 4^2 + 3^2 = 25 = 5^2 \quad (12)$$

And if $x_3 = 12$, then

$$5^2 + 12^2 = 169 = 13^2 \leftrightarrow 5^2 = 13^2 - 12^2 \quad (13)$$

Substitute (13) to (12) to obtain the (14):

$$4^2 + 3^2 = 13^2 - 12^2 \leftrightarrow 4^2 + 3^2 + 12^2 = 13^2 \quad (14)$$

And, if $\mathrm{x}_4 = 84$, then

$$13^2 + 84^2 = 7225 = 85^2 \leftrightarrow 13^2 = 85^2 - 84^2 \quad (15)$$

Then, substitute (15) into (14) to obtain the (16) as shown below:

$$4^2 + 3^2 + 12^2 = 85^2 - 84^2 \leftrightarrow$$
$$4^2 + 3^2 + 12^2 + 84^2 = 85^2 \quad (16)$$

3.2 Programming the Expansion Intuition

To apply the expansion intuition, generalize it, and employ it in various fields, it will be programmed in the Python language [21], where Figure 3 shows the action mechanism for the new intuition, which facilitates the program construction to it as follows.
Figure 3 shows the programming for action mechanism of the new intuition as follows:

$$x_1^2 + x_2^2 = h_1^2, \text{then } h_1^2 + x_3^2 = h_2^2, \text{then } h_2^2 + x_4^2 h_3^2, \text{then } h_3^2 + x_5^2 = h_4^2, \text{ then}, \cdots, h_{k-2}^2 + x_k^2 = h_{k-1}^2 \quad (17)$$

This can be explained through the flow chart as shown in Figure 4.

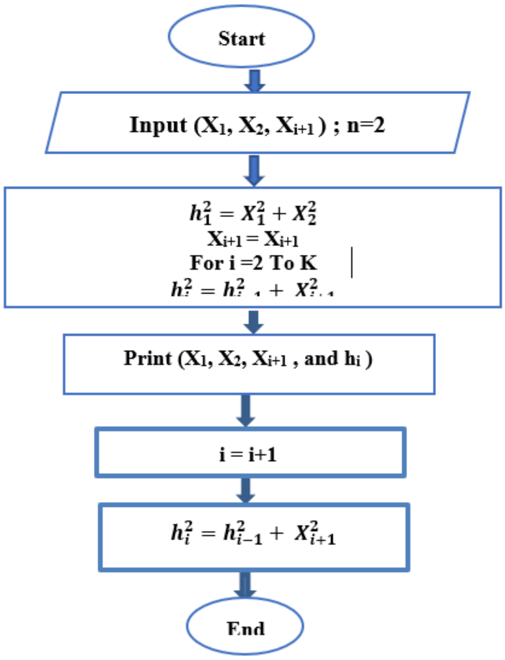

Figure 4: Flowchart for Abd's 1st intuition in k-dimension.

Therefore, we can express the programming according to the following steps:

Step 1: Sides of a right triangle (x_1, x_2) and the number of iteration to generate a right triangle (k)
Step 2: h_1 < ---- $x_1^2 + x_2^2$
Step 3: Loop of range (k)
Step 4: x_{i+2} < -----random number
Step 5: $h_{i+1}^2 = X_{i+1}^2 + h_i^2$
Step 6: Display h_{i+1} , X_1 , X_2 , X_{i+1}, h_i

3.3 Mathematical Induction to Prove the Intuition

Let us restate our finding in mathematical statement and prove it using the method of mathematical induction. Recall that we construct k right triangles beginning with the foundation based right triangle with legs x_1, x_2 and hypotenuse h_1. The next triangle is another right triangle where one leg is h_1 and the other is a new value, say x_3. The hypotenuse of this right triangle is denoted as h_2. We then construct another right triangle where one leg is h_2 and the other is another new value, x_4. We repeat this procedure to contruct k right triangles. Observe that $(k-1)$-th and k-th triangles share the side, h_{k-1}. We found that $x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = h_k^2$, as stated by the following theorem and its proof.

Theorem 3.1. Consider a sequence of k right triangles constructed as follows:

The first right triangle has legs x_1, and x_2, with hypotenuse h_1. For each $k \in \mathbb{N}\backslash\{1\}$, the k-th triangle has legs x_{k+1} and h_{k-1}, with hypotenuse h_k.We found that $x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = h_k^2$.

Proof: we are going to prove this result by induction method.

For the case $n = 1$, we consider the first right triangle, and Pythagoras Theorem gives that $x_1^2 + x_2^2 = h_1^2$. So the first case is proven.

We assume that, it is true for the $n = k$, i.e.

$$x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = h_k^2 \tag{18}$$

So now, we aim to prove the statement for $n = k + 1$.

By adding (x_{k+2}^2) to both sides for (13), we get

$$x_1^2 + x_2^2 + \cdots + x_{k+1}^2 + x_{k+2}^2 = h_k^2 + x_{k+2}^2 \tag{19}$$

Proof: we firstly claim that $x_1 = h_k \cos^k \theta$ and prove it by induction method. For the case $k = 1$, it is obvious that $x_1 = h_1 \cos\theta$, since θ is the acute angle adjacent to x_2. We assume that it is also true for the case $k = n$, which is

$$x_1 = h_n \cos^n \theta \tag{20}$$

Now let us consider the $(k+1)$-th right trianlge as Figure 5.

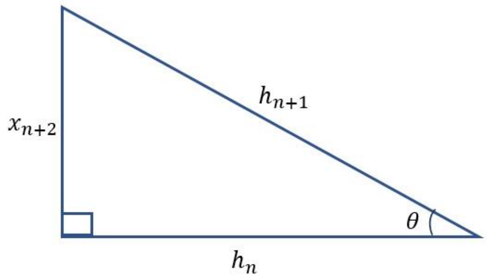

Figure 5: The $(k+1)$-th right trianlge.

Therefore

$$h_n = h_{n+1} \cos\theta \tag{21}$$

Substitute (21) into (20), we then get on (22) as:

$$x_1 = (h_{n+1}\cos\theta)(\cos^n\theta) = h_{n+1}\cos^{n+1}\theta \tag{22}$$

Therefore it is true that $x_1 = h_k \cos^k \theta$.

Now, for the sequence of $\{h_1, h_2, \cdots, h_k\}$ let us calculate it rate of growth $\frac{h_i}{h_{i+1}}$.

Since $h_i = \frac{x_1}{\cos^i\theta}$ and $h_{i+1} = \frac{x_1}{\cos^{i+1}\theta}$, then we found that

$$\frac{h_i}{h_{i+1}} = \cos\theta. \tag{23}$$

Since $0 < \theta < \frac{\pi}{2}$, then $0 < \cos\theta < 1$ and this shows that the sequence $\{h_1, h_2, \cdots, h_k\}$ is increasing with sub-exponential growth, since the terms h_i's are increasing but at a progressively slower rate.

Now, we are going to consider another special case which is the right triangle with equal legs (isosceles right triangle). We then are able to determine the hyphotenus of each k-th isosceles right triangle.

Proposition 3.3: consider a sequence of k isosceles right triangles constructed as follows:

The first right triangle has two legs x_1 and hypotenuse h_1. For each $k \in \mathbb{N}\backslash\{1\}$, the k-th triangle has two legs h_{k-1}, and hypotenuse h_k.

Therefore,

$$h_k = \begin{cases} 2^n x_1 & \text{if } k = 2n \text{ for an integer } n \\ \frac{2^n x_1}{\sqrt{2}} & \text{if } k = 2n - 1 \text{ for an integer } n \end{cases} \tag{24}$$

Proof: let us consider the first and second isosceles right triangles and applying the basic trigonometry, we then have $h_1 = \frac{2x_1}{\sqrt{2}}$ and $h_2 = 2x_1$, then the finding is true for the case $k = 1$ and $k = 2$. By induction we assume that

$$h_k = \begin{cases} Kx_1 & \text{if } k = 2n \text{ for an integer } n \\ \frac{(K+1)x_1}{\sqrt{2}} & \text{if } k = 2n - 1 \text{ for an integer } n \end{cases} \tag{25}$$

for an integer k.

To prove for the case $k + 1$, we consider two cases. The firsrt case is whenever $k + 1 = 2n$ for an integer n and we consider the $(k+1)$-th triangle.

Therefore, $h_{k+1} = \frac{2h_k}{\sqrt{2}}$. Since $k + 1 = 2n$, then $k = 2n - 1$ and we have $h_k = \frac{2^n x_1}{\sqrt{2}}$. Therefore

$$h_{k+1} = \frac{2}{\sqrt{2}}\left(\frac{2^n x_1}{\sqrt{2}}\right) = 2^n x_1.$$

Now for the second case, we let $k + 1 = 2n - 1$ for an integer n. Therefore $k = 2n - 2 = 2m$ for some other integer $m = n - 1$. Hence $h_k = 2^m x_1 = 2^{n-1} x_1$. Then $h_{k+1} = \frac{2h_k}{\sqrt{2}} = \frac{2}{\sqrt{2}}(2^{n-1} x_1) = \frac{2^n x_1}{\sqrt{2}}$, and the proof is achieved.

4 CONCLUSIONS

Our study has led to a new creation of a modern expansion theorem derived from the Pythagorean Theorem (PT), viable in multi planes and dimensions. This extension confirms that the square of the hypotenuse of a right-angled triangle, created by integrating several different right triangles in space, equals the sum of the squares legs of all these triangles. Via projecting the Pythagorean rule to higher dimensions and adding a new dimension with each transition, this intuition was verified utilizing the rule of perfect squares and proved it via the mathematical induction process.

This new widened theorem shows a more overall process than the conventional PT applied on two dimensions only. Therefore, it can be utilized in varied fields including network security, coding, wireless communications networks, encryption, IoT (Internet of Things), and AI (Artificial Intelligence). This stratifies with the rapid progress in Information and Communication Technology (ICT) and gives valuable insights for applications via these scopes.

The improvement of PT via expanding to multiple dimensions has wide and influential applications. In communications, it improves signal processing and network design through advanced multidimensional analysis. Also, it can improve encryption algorithms and error detection with correcting it by using multidimensional spaces to protect data in the field of security and encryption. In artificial intelligence (AI), it supports and improves modern machine learning algorithms and techniques. It also helps in integrating sensors, analyzing data, and improving accuracy and functionality in the Internet of Things (IoT). In addition, this expansion helps in quantum computing and structural analysis, providing more accurate solutions and models in the fields of engineering and physics. These extensions are also invested in virtual reality experiences and improving 3D modeling of computer graphics and visualization. In general, this extension keeps pace with all modern and accelerating technological techniques across many and diverse fields. Finally, we recommend investing in this study and using it in the Network Security, Artificial Intelligence, Aerospace Engineering, and Mathematical and Physical applications.

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