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Fredholm property of non-smooth pseudodifferential operators

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Abstract
In this paper we prove sufficient conditions for the Fredholm property of a non-smooth pseudodifferential operator P which symbol is in a Hölder space with respect to the spatial variable. As a main ingredient for the proof we use a suitable symbol-smoothing.

KEYWORDS
Fredholm property, non-smooth pseudodifferential operators

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1 | INTRODUCTION

Fredholm operators are often called nearly invertible operators. They admit dimension formulae similar to linear operators between finite dimensional spaces. Because of this they play an important role in the field of partial differential equations in order to get existence and uniqueness results. Great effort was already spent to get conditions for the Fredholmness of smooth pseudodifferential operators with symbols in the Hörmander-class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n) := \bigcap_{M\in\mathbb{N}} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, where $0 \le \rho, \delta \le 1$ and $m \in \mathbb{R}$. Here the symbol-class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ consists of all M-times continuous differentiable functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ that are smooth with respect to the spatial variable such that for all $k \in \mathbb{N}_0$

$$|a|^{(m)}_k := \max_{|\alpha|\le\min\{k,M\},|\beta|\le k} \sup_{x,\xi\in\mathbb{R}^n} \left|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| \langle\xi\rangle^{-(m-\rho|\alpha|+\delta|\beta|)} < \infty.$$

For every symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we define the associated pseudodifferential operator via

$$OP(a)u(x) := a\big(x, D_x\big)u(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\, đ\xi \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n, \tag{1.1}$$

where $đ\xi := \frac{1}{(2\pi)^n} d\xi$ and $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space, i.e., the space of all rapidly decreasing smooth functions and $\hat{u}$ is the Fourier transformation of u.

In [9] Kohn and Nirenberg showed, that the ellipticity of a classical smooth pseudodifferential operator is necessary for its Fredholm property. Apart from necessary conditions Kumano-go gave in [11, Chapter III, Theorem 5.16] sufficient conditions for the Fredholmness of smooth pseudodifferential operators. He showed that pseudodifferential operators with so called *slowly varying* smooth symbols a of order m are Fredholm operators from $H^m_2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ under certain ellipticity conditions. The ellipticity conditions are satisfied if a is uniformly elliptic in the sense that

$$|a(x,\xi)| \geq C|\xi|^m \qquad \text{for all } x,\xi \in \mathbb{R}^n \text{ with } |x|+|\xi| \geq R \tag{1.2}$$

for some $R, C > 0$. Here $H^s_p(\mathbb{R}^n)$ denotes a Bessel Potential Space for $p \in (1,\infty)$ and $s \in \mathbb{R}$, defined in Section 2. Moreover, $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is slowly varying if for all $\alpha, \beta \in \mathbb{N}^n_0$ with $\beta \neq 0$ we have

$$\left|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| \leq C_{\alpha,\beta}(x)\big(1+|\xi|\big)^{m+\delta|\beta|-\rho|\alpha|}$$

for a bounded function $C_{\alpha,\beta} : \mathbb{R}^n \to \mathbb{R}$ with $C_{\alpha,\beta}(x) \to 0$ if $|x| \to \infty$. In [18] Schrohe extended the result of Kumano-go as follows: Smooth pseudodifferential operators with slowly varying symbols of the order zero are Fredholm operators on the weighted Sobolev spaces $H^{st}_\gamma(\mathbb{R}^n)$, see [18] for the definition, if and only if its symbol is uniformly elliptic.

In applications (e.g. to non-linear PDEs) also non-smooth pseudodifferential operators appear naturally. Therefore we are interested in sufficient conditions for non-smooth pseudodifferential operators to become a Fredholm operator from $H^m_p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, where $m \in \mathbb{R}$, $1 < p < \infty$. For non-smooth differential operators the Fredholm property can be characterized by the uniform ellipticity of its symbol. This was announced by Cordes in [3], completed by Illner in [8] and partially recovered by Fan and Wong in [5]. This characterization of the Fredholm property was extended to the matrix-valued case in [6] for $p = 2$ and in [19] for general $p \in (1,\infty)$. In the case $p = 2$ an alternative proof by means of the tool of C^*-algebras, was given by Taylor in [20]. The goal of this paper is to give sufficient conditions for the Fredholm property of pseudodifferential operators $a\big(x, D_x\big)$ with a symbol a in the non-smooth symbol-class $C^{\tilde{m},\tau} S^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\big(\mathbb{C}^N\big)\big)$, $0 \leq \rho, \delta \leq 1$, $M \in \mathbb{N}_0 \cup \{\infty\}$, $m \in \mathbb{R}$. For the definition of the Hölder space $C^{\tilde{m},\tau}$ of the order $\tilde{m} \in \mathbb{N}_0$ with Hölder regularity $0 < \tau \leq 1$ we refer to Section 2 below. A function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is an element of the symbol-class $C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, $m \in \mathbb{R}$, if the following properties hold for all $\alpha, \beta \in \mathbb{N}^n_0$ with $|\beta| \leq \tilde{m}$ and $|\alpha| \leq M$:

i) $\partial^\beta_x a(x,.) \in C^M(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,
ii) $\partial^\beta_x \partial^\alpha_\xi a \in C^0\big(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\big)$,
iii) $\left|\partial^\alpha_\xi a(x,\xi)\right| \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}$ for all $x, \xi \in \mathbb{R}^n$,
iv) $\left\|\partial^\alpha_\xi a(.,\xi)\right\|_{C^{\tilde{m},\tau}(\mathbb{R}^n)} \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|+\delta(\tilde{m}+\tau)}$ for all $\xi \in \mathbb{R}^n$.

Moreover, $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathscr{L}\big(\mathbb{C}^N\big)$ is an element of the symbol-class $C^{\tilde{m},\tau} S^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\big(\mathbb{C}^N\big)\big)$, $m \in \mathbb{R}$, $N \in \mathbb{N}$, if and only if $a_{j,k} \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for all $j, k = 1, \ldots, N$, where we identify $A \in \mathscr{L}\big(\mathbb{C}^N\big)$ with a matrix $\big(a_{j,k}\big)^N_{j,k=1} \in \mathbb{C}^{N\times N}$ in the standard way. For a given symbol a we define the associated pseudodifferential operator as in the smooth case, cf. (1.1). We remark that in the literature there are also some results concerning the Fredholm property of pseudodifferential operators on compact manifolds, see e.g. [7], [15]. Nistor even gave some criteria for the Fredholmness of pseudodifferential operators on non-compact manifolds in [16].

In the present paper we proceed as follows: We give a short summary of all notations and function spaces needed in Section 2. Moreover we introduce the space of amplitudes and the oscillatory integrals. In Section 3 we define all symbol-classes of pseudodifferential operators needed later on and present their properties. In particular we extend the concept of symbol-smoothing presented in [21, Section 1.3]. Together with the extension of the symbol reduction result of [2] for non-smooth double symbols, see Subsection 3.2 below, the symbol-smoothing becomes the main ingredient in order to verify the main result of our paper:

Theorem 1.1. *Let $\tilde{m}, N \in \mathbb{N}$, $0 < \tau < 1$, $0 \leq \delta < \rho \leq 1$, $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $p \in (1,\infty)$ with $p = 2$ if $\rho \neq 1$. Additionally we choose an arbitrary $\theta \in (0; \min\{(\tilde{m}+\tau)(\rho-\delta); 1\})$ and $\tilde{\varepsilon} \in (0, \min\{(\rho-\delta)\tau; (\rho-\delta)(\tilde{m}+\tau)-\theta; \theta)\})$. Moreover let $a \in C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\big(\mathbb{C}^N\big)\big)$ be a symbol fulfilling the following properties for some $R > 0$ and $C_0 > 0$:*

1) $|\det(a(x,\xi))|\langle\xi\rangle^{-mN} \geq C_0$ for all $x, \xi \in \mathbb{R}^n$ with $|x| + |\xi| \geq R$.
2) $a(x,\xi) \xrightarrow{|x|\to\infty} a(\infty,\xi)$ for all $\xi \in \mathbb{R}^n$.

Then for all $M \geq (n+2) + n \cdot \max\{1/2, 1/p\}$ and $s \in \mathbb{R}$ with

$$(1-\rho)\frac{n}{2} - (1-\delta)(\tilde{m}+\tau) + \theta + \tilde{\varepsilon} < s < \tilde{m}+\tau$$

the operator

$$a(x, D_x) : H_p^{m+s}(\mathbb{R}^n)^N \to H_p^s(\mathbb{R}^n)^N$$

is a Fredholm operator.

As in the smooth case, we restrict ourselves to the case of slowly varying symbols in order to show the Fredholm property. As Schrohe already wrote in [18] for a parameter construction of non-classical smooth symbols more than invertibility of the symbol is needed and the parametrix can differ from the Fredholm inverse. We see, that many conditions are needed in Theorem 1.1 to show the Fredholm property of a non-smooth pseudodifferential operator. Hence the question arises which of them are of technical nature and which of them are really necessary. In the smooth case Schrohe showed in [18] that the uniform ellipticity of a zero order symbol a is a necessary condition for $a(x, D_x)$ being a Fredholm operator. By means of the composition with order reducing operators one easily obtains that the uniform ellipticity of a smooth symbol a of arbitrary order is also a necessary condition for $a(x, D_x)$ being a Fredholm operator. Uniform ellipticity for systems is equivalent to condition 1). For non-smooth differential operators this condition is also necessary, cf. [19]. Therefore 1) is necessary at least if a is smooth or $a(x, D_x)$ is a differential operator. Additionally in the smooth case, also the condition $0 \leq \delta < \rho \leq 1$ arises. Since each Fredholm operator $T : H_p^{m+s}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$ is continuous, it is natural to impose

i) $M > \max\left\{\frac{n}{2}, \frac{n}{p}\right\}$,
ii) $\tilde{m} + \tau > \frac{1-\rho}{1-\delta} \cdot \frac{n}{2}$ if $\rho < 1$ and $\tilde{m} + \tau > 0$ if $\rho = 1$ respectively,
iii) $(1-\rho)\frac{n}{p} - (1-\delta)(\tilde{m} + \tau) < s < \tilde{m} + \tau$

in order to apply the known results on mapping properties of non-smooth pseudodifferential operators. In order to prove the claim of Theorem 1.1, we need to strengthen condition *iii*) due to technical reasons. Finally, also condition 2) is of technical nature.

Theorem 1.1 will be proved in Section 4. For the definition of the symbol-class $C^{\tilde{m},\tau}\tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}(\mathbb{C}^N))$ we refer to Definition 3.5 in Subsection 3.1.

2 | NOTATIONS, DEFINITIONS AND FUNCTION SPACES

The set of all natural numbers without 0 is denoted by $\mathbb{N}$. Unless otherwise noted we consider $n \in \mathbb{N}$ during the whole paper. We define

$$\langle x \rangle := \left(1 + |x|^2\right)^{1/2} \quad \text{for each } x \in \mathbb{R}^n \qquad \text{and} \qquad đ\xi := (2\pi)^{-n} d\xi.$$

Moreover

$$\langle x; y \rangle := \left(1 + |x|^2 + |y|^2\right)^{1/2} \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Additionally we set for each $x \in \mathbb{R}$

$$\lfloor x \rfloor := \max\{l \in \mathbb{Z} : l \leq x\} \qquad \text{and} \qquad \lceil x \rceil := \min\{l \in \mathbb{Z} : l \geq x\}.$$

For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we use the notations $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$ and $D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha$.

Assuming two Banach spaces X, Y the set of all linear and bounded operators $A : X \to Y$ is denoted by $\mathscr{L}(X, Y)$. In case $X = Y$, we just write $\mathscr{L}(X)$.

For $s \in (0, 1]$ the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ fulfilling

$$\|f\|_{C^{0,s}} \equiv \|f\|_{C^{0,s}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^s} < \infty$$

is called *Hölder space* $C^{0,s}(\mathbb{R}^n)$ of the order 0 with Hölder continuity exponent s. A function $f : \mathbb{R}^n \to \mathbb{C}$ is in the Hölder space $C^{\tilde{m},s}(\mathbb{R}^n)$ of the order $\tilde{m} \in \mathbb{N}_0$ if we have $\partial_x^\alpha f \in C^{0,s}(\mathbb{R}^n)$ for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{m}$. Note that all Hölder spaces are Banach spaces.

On account of the definition of the Hölder spaces and the Leibniz-rule we obtain:

Lemma 2.1. *Let $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$ and $f, g \in C^{\tilde{m},\tau}(\mathbb{R}^n)$. Then*

$$\|fg\|_{C^{\tilde{m},\tau}} \leq \sum_{\tilde{m}_1+\tilde{m}_2=\tilde{m}} C_{\tilde{m}} \left\{ \|f\|_{C_b^{\tilde{m}_1}} \|g\|_{C^{\tilde{m}_2,\tau}} + \|f\|_{C^{\tilde{m}_1,\tau}} \|g\|_{C_b^{\tilde{m}_2}} \right\}.$$

The *Bessel Potential space* $H^s_p(\mathbb{R}^n)$, $s \in \mathbb{R}$ and $1 < p < \infty$, will play a central role in this paper. The set $H^s_p(\mathbb{R}^n)$ is defined by

$$H^s_p(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x \rangle^s f \in L^p(\mathbb{R}^n) \right\}, \tag{2.1}$$

where $\langle D_x \rangle^s := OP\big(\langle \xi \rangle^s\big)$.

For the convenience of the reader we mention an interpolation result needed in this paper:

Lemma 2.2. *Let $k, m \in \mathbb{N}$ with $k \leq m$, $0 < \tau < 1$ and $\theta := \frac{k}{m+\tau}$. Then*

$$\|f\|_{C_b^k(\mathbb{R}^n)} \leq C \|f\|^{1-\theta}_{C_b^0(\mathbb{R}^n)} \|f\|^{\theta}_{C^{m,\tau}(\mathbb{R}^n)} \qquad \text{for all } f \in C^{m,\tau}(\mathbb{R}^n).$$

Proof. For all $p \in [1, \infty]$ we denote the real interpolation spaces by $\big(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n)\big)_{\theta,p}$, cf. e.g. [12]. An application of the reiteration theorem, c.f. [13, Theorem 1.2.15], and of Proposition 1.20 in [12] provides

$$\big(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n)\big)_{\theta,1} \subseteq \big(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n)\big)_{\frac{k}{m+1},1} \subseteq C_b^k(\mathbb{R}^n).$$

This yields the claim. For more details we refer to [17, Lemma 2.41]. □

Since this paper deals with the Fredholm property of pseudodifferential-operators, we finally add the definition of an Fredholm operator:

Definition 2.3. Let X, Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$. Then T is called a *Fredholm operator* if $\mathscr{N}(T)$ is finite dimensional and $\mathscr{R}(T)$ is closed and has finite co-dimension, i.e., there is a finite dimensional subspace $Z \subseteq Y$ such that $Y = \mathscr{R}(T) \oplus Z$.

The following characterization is fundamental for our purposes.

Theorem 2.4. *Let X, Y be Banach spaces. Then $T \in \mathscr{L}(X, Y)$ is a Fredholm operator if and only if there are some operators $B, C \in \mathscr{L}(X, Y)$ and some compact operators $K_1 \in \mathscr{L}(X)$, $K_2 \in \mathscr{L}(Y)$ such that*

$$BT = I_X - K_1 \qquad TC = I_Y - K_2,$$

where I_X respectively I_Y are the identity operators on X respectively Y.

The proof can e.g. be found in [4, Theorem 3.15].

2.1 | Space of amplitudes and oscillatory integrals

The aim of the present paper is to define and discuss some properties of oscillatory integrals for all elements of the space of amplitudes $\mathscr{A}^{m,N}_{\tau,M}(\mathbb{R}^n \times \mathbb{R}^n)$. Here $\mathscr{A}^{m,N}_{\tau,M}(\mathbb{R}^n \times \mathbb{R}^n)$, $N, M \in \mathbb{N}_0 \cup \{\infty\}$, $m, \tau \in \mathbb{R}$ is the set of all continuous functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$, $|\beta| \leq M$ we have

i) $\partial_\eta^\alpha \partial_y^\beta a(y, \eta) \in C^0\big(\mathbb{R}^n_y \times \mathbb{R}^n_\eta\big)$,

ii) $\left|\partial_\eta^\alpha \partial_y^\beta a(y, \eta)\right| \leq C_{\alpha,\beta}(1 + |\eta|)^m (1 + |y|)^\tau$ for all $y, \eta \in \mathbb{R}^n$,

where all derivatives are well defined in the sense of distributions. For all elements $a \in \mathscr{A}^{m,N}_{\tau,M}(\mathbb{R}^n \times \mathbb{R}^n)$, $N, M \in \mathbb{N}_0 \cup \{\infty\}$, $m, \tau \in \mathbb{R}$ the oscillatory integral is defined by

$$\text{Os-}\iint e^{-iy\cdot\eta} a(y, \eta)\, dy\, đ\eta := \lim_{\varepsilon \to 0} \iint \chi(\varepsilon y, \varepsilon \eta) e^{-iy\cdot\eta} a(y, \eta)\, dy\, đ\eta, \tag{2.2}$$

where $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0, 0) = 1$.

Defining for all $m \in \mathbb{N}$

$$A^m(D_x,\xi) := \langle\xi\rangle^{-m}\langle D_x\rangle^m \qquad \text{if } m \text{ is even,} \tag{2.3}$$

$$A^m(D_x,\xi) := \langle\xi\rangle^{-m-1}\langle D_x\rangle^{m-1} - \sum_{j=1}^{n}\langle\xi\rangle^{-m}\frac{\xi_j}{\langle\xi\rangle}\langle D_x\rangle^{m-1}D_{x_j} \qquad \text{else,} \tag{2.4}$$

we can extend some properties of the oscillatory integral proved in Section 2.3 of [2] as follows:

Theorem 2.5. *Let $m,\tau \in \mathbb{R}$ and $N, M \in \mathbb{N}_0 \cup \{\infty\}$ with $N > n+\tau$. Moreover let $l, l' \in \mathbb{N}$ with $N \geq l' > n+\tau$ and $M \geq l > n+m$. Then the oscillatory integral (2.2) exists for all $a \in \mathscr{A}^{m,N}_{\tau,M}(\mathbb{R}^n \times \mathbb{R}^n)$ and we have for all $l_1, l_2 \in \mathbb{N}_0$ with $l_1 \leq N$ and $l_2 \leq l$:*

$$Os\text{-}\iint e^{-iy\cdot\eta}a(y,\eta)\,dy\,đ\eta = \iint e^{-iy\cdot\eta}A^{l'}(D_\eta,y)A^l(D_y,\eta)a(y,\eta)\,dy\,đ\eta,$$

$$Os\text{-}\iint e^{-iy\cdot\eta}a(y,\eta)\,dy\,đ\eta = Os\text{-}\iint e^{-iy\cdot\eta}A^{l_1}(D_\eta,y)A^{l_2}(D_y,\eta)a(y,\eta)\,dy\,đ\eta.$$

Proof. The claim can be verified in the same way as in Theorem 2.10 and Theorem 2.12 of [2], if one takes care of ii) just holding for $|\beta| \leq l$. □

Theorem 2.6. *Let $m,\tau \in \mathbb{R}$, $m_i, \tau_i \in \mathbb{R}$ for $i \in \{1,2\}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ such that there is a $l' \in \mathbb{N}$ with $N \geq l' > n+\tau$. Moreover let $\alpha,\beta \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{M}$, where $\tilde{M} := \max\{\hat{m} \in \mathbb{N}_0 : N - \hat{m} > n+\tau\}$ and $l \in \mathbb{N}$ with $l > m+n$. Considering $a \in C^0(\mathbb{R}^n_y \times \mathbb{R}^n_{y'} \times \mathbb{R}^n_\eta \times \mathbb{R}^n_\xi)$ with*

- $\left|A^{l'}(D_\eta,y)A^l(D_y,\eta)a(y,y',\eta,\xi)\right| \leq C_{l,l'}\langle y\rangle^{\tau-l'}\langle\eta\rangle^{m-l}\langle y'\rangle^{\tau_1}\langle\xi\rangle^{m_1}$,
- $\left|A^{l'}(D_\eta,y)A^l(D_y,\eta)\partial_\xi^\alpha\partial_{y'}^\beta a(y,y',\eta,\xi)\right| \leq C_{l,l',\alpha,\beta}\langle y\rangle^{\tau-l'}\langle\eta\rangle^{m-l}\langle y'\rangle^{\tau_2}\langle\xi\rangle^{m_2}$

for all $y, y', \eta, \xi \in \mathbb{R}^n$ we have for all $y', \xi \in \mathbb{R}^n$:

$$\partial_\xi^\alpha\partial_{y'}^\beta Os\text{-}\iint e^{-iy\cdot\eta}a(y,y',\eta,\xi)\,dy\,đ\eta = Os\text{-}\iint e^{-iy\cdot\eta}\partial_\xi^\alpha\partial_{y'}^\beta a(y,y',\eta,\xi)\,dy\,đ\eta.$$

Proof. This result can be verified similarly to [2, Theorem 2.11]. □

Corollary 2.7. *Let $m,\tau \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ such that there is some $l' \in \mathbb{N}$ with $N \geq l' > n+\tau$. Moreover let $l \in \mathbb{N}$ with $l > n+m$. Additionally let $a_j, a \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$, $j \in \mathbb{N}_0$ such that for all $\alpha,\beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $|\beta| \leq l$ the derivatives $\partial_\eta^\alpha\partial_y^\beta a_j, \partial_\eta^\alpha\partial_y^\beta a$ exist in the classical sense and*

- $\left|\partial_\eta^\alpha\partial_y^\beta a_j(y,\eta)\right| \leq C_{\alpha,\beta}\langle\eta\rangle^m\langle y\rangle^\tau$ *for all* $\eta, y \in \mathbb{R}^n$, $j \in \mathbb{N}_0$,
- $\left|\partial_\eta^\alpha\partial_y^\beta a\right| \leq C_{\alpha,\beta}\langle\eta\rangle^m\langle y\rangle^\tau$ *for all* $\eta, y \in \mathbb{R}^n$,
- $\partial_\eta^\alpha\partial_y^\beta a_j(y,\eta) \xrightarrow{j\to\infty} \partial_\eta^\alpha\partial_y^\beta a(y,\eta)$ *for all* $\eta, y \in \mathbb{R}^n$.

Then

$$\lim_{j\to\infty} Os\text{-}\iint e^{-iy\cdot\eta}a_j(y,\eta)\,dy\,đ\eta = Os\text{-}\iint e^{-iy\cdot\eta}a(y,\eta)\,dy\,đ\eta.$$

Proof. The claim can be shown similarly to [2, Corollary 2.13]. □

Another property of oscillatory integral needed later on is:

Remark 2.8. Assuming $u \in C_b^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we obtain

$$\text{Os}\text{-}\iint e^{i(x-y)\cdot\eta}u(y)\,dy\,đ\eta = u(x).$$

For the proof see e.g. [1, Example 3.11].

3 | PSEUDODIFFERENTIAL OPERATORS AND THEIR PROPERTIES

Throughout this section we summarize all properties of pseudodifferential operators needed later on. Additionally we define all symbol-classes of pseudodifferential operators needed in this paper.

On account of Lemma 2.2 with $\theta := \frac{\tilde{m}+s}{\tilde{m}+1}$ if $s < 1$ and by means of $C_b^{k+1}(\mathbb{R}^n) \subseteq C^{k,1}(\mathbb{R}^n)$ else we can show

$$S^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M\big) \subseteq C^{\tilde{m},s} S^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M\big), \tag{3.1}$$

for all $0 < s \leq 1$, $\tilde{m} \in \mathbb{N}_0$, $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho, \delta \leq 1$.

For more details see, [17, Remark 4.2].

Additionally we get by means of interpolation, c.f. Lemma 2.2, the next estimate for non-smooth symbols:

Remark 3.1. Let $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$, $0 \leq \delta, \rho \leq 1$, $m \in \mathbb{R}$ and $a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M\big)$. Then we get for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $k \in \mathbb{N}_0$ with $k \leq \tilde{m}$:

$$\left\| \partial_\xi^\alpha a(.,\xi) \right\|_{C_b^k(\mathbb{R}^n)} \leq C_\alpha \langle \xi \rangle^{m-\rho|\alpha|+\delta k} \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Pseudodifferential operators are bounded as maps between several Bessel Potential spaces. For the proof we refer to [2, Theorem 3.7].

Theorem 3.2. *Let $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ with $\rho > 0$, $1 < p < \infty$ and $M \in \mathbb{N}_0 \cup \{\infty\}$ with $M > max\left\{\frac{n}{2}, \frac{n}{p}\right\}$. Additionally let $\tilde{m} \in \mathbb{N}_0$ and $0 < \tau \leq 1$ such that $\tilde{m} + \tau > \frac{1-\rho}{1-\delta} \cdot \frac{n}{2}$ if $\rho < 1$ and $\tilde{m} \in \mathbb{N}_0$, $\tau > 0$ if $\rho = 1$ respectively. Moreover let $\mathscr{B} \subseteq C^{\tilde{m},\tau} S^{m-k_p}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be bounded with $k_p := (1-\rho)n\,|1/2 - 1/p|$ and let $(1-\rho)n/p - (1-\delta)(\tilde{m}+\tau) < s < \tilde{m}+\tau$. Then there is some $C_s > 0$, independent of $a \in \mathscr{B}$, such that*

$$\left\| a\big(x, D_x\big) f \right\|_{H_p^s(\mathbb{R}^n)} \leq C_s \| f \|_{H_p^{s+m}(\mathbb{R}^n)} \qquad \text{for all } a \in \mathscr{B} \text{ and } f \in H_p^{s+m}(\mathbb{R}^n).$$

3.1 | Symbol-smoothing

A well-known tool for proving some properties of non-smooth pseudodifferential operators of the symbol class $XS^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for certain Banach spaces X is the symbol-smoothing, see e.g. [21, Section 1.3]. In order to prove the Fredholm property of non-smooth pseudodifferential operators, we now generalize the tool of symbol-smoothing for pseudodifferential operators which are non-smooth with respect to the second variable and for $\rho \neq 1$. To this end we fix two functions $\phi, \psi_0 \in C_0^\infty(\mathbb{R}^n)$ till the end of this section with the following properties:

- $\phi(\xi) = 1$ for all $|\xi| \leq 1$,
- $\psi_0 \geq 0$, $\psi_0(\xi) = 1$ for all $|\xi| \leq 1$ and $\psi_0(\xi) = 0$ for all $|\xi| \geq 2$.

Then we define for all $j \in \mathbb{N}$ the functions ψ_j via

$$\psi_j(\xi) := \psi_0\big(2^{-j}\xi\big) - \psi_0\big(2^{-j-1}\xi\big) \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Using that for any $a \in \mathbb{R}$ there are $C_1, C_2 > 0$ such that

$$C_1 \langle \xi \rangle^{-a} \leq 2^{-ja} \leq C_2 \langle \xi \rangle^{-a} \qquad \text{for all } \xi \in \text{supp}\,\big(\psi_j\big), j \in \mathbb{N}, \tag{3.2}$$

we can show the following properties of the functions ψ_j for all $\alpha \in \mathbb{N}_0^n$:

$$\left\| \partial_\xi^\alpha \psi_j \right\|_\infty \leq C_\alpha \langle \xi \rangle^{-|\alpha|}. \tag{3.3}$$

Additionally we define for all $\varepsilon > 0$ the operator J_ε by

$$J_\varepsilon := \phi\big(\varepsilon D_x\big).$$

Note, that for each $\alpha \in \mathbb{N}_0^n$:

$$\partial_\xi^\alpha J_\varepsilon = J_\varepsilon \partial_\xi^\alpha. \tag{3.4}$$

The operator J_ε has the following properties:

Lemma 3.3. *For $\varepsilon > 0$, $0 < \tau < 1$ and $\tilde{m} \in \mathbb{N}_0$ we have for all $f \in C^{\tilde{m},\tau}(\mathbb{R}^n)$:*

i) $\left\| D_x^\beta J_\varepsilon f \right\|_\infty \le C \|f\|_{C^{\tilde{m},\tau}(\mathbb{R}^n)}$ *for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \tilde{m}$.*

ii) $\left\| D_x^\beta J_\varepsilon f \right\|_\infty \le C \varepsilon^{-(|\beta| - \tilde{m} - \tau)} \|f\|_{C^{\tilde{m},\tau}(\mathbb{R}^n)}$ *for all $\beta \in \mathbb{N}_0^n$ with $|\beta| > \tilde{m}$.*

iii) Let $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \tilde{m}$ and $t \ge 0$ with $\tilde{m} + \tau - t - |\beta| > 0$ and $\tilde{m} + \tau - t - |\beta| \notin \mathbb{N}$. Then we have for $\tilde{m}_1 \in \mathbb{N}_0$ and $0 < s < 1$ with $\tilde{m}_1 + s = \tilde{m} + \tau - |\beta| - t$:

$$\left\| D_x^\beta (1 - J_\varepsilon) f \right\|_{C^{\tilde{m}_1,s}(\mathbb{R}^n)} \le C \varepsilon^t \|f\|_{C^{\tilde{m},\tau}(\mathbb{R}^n)}.$$

iv) $\left\| D_x^\beta (1 - J_\varepsilon) f \right\|_\infty \le C_{\tilde{m},\tau} \varepsilon^{\tilde{m}+\tau-|\beta|} \left\| D_x^\beta f \right\|_{C^{\tilde{m}-|\beta|,\tau}(\mathbb{R}^n)}$ *for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \tilde{m}$.*

Proof. On account of [21, Lemma 1.3C] the claims i), ii) and claim iv) in the case $|\beta| = 0$ hold true. An application of the case $|\beta| = 0$ on $g := D_x^\beta f \in C^{\tilde{m}-|\beta|,\tau}(\mathbb{R}^n)$ provides the general case of claim iv). Because of [21, Lemma 1.3.A] we additionally obtain claim iii) for the case $|\beta| = 0$. It remains to verify claim iii) for general $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \tilde{m}$. This can be done similarly to the proof of the case $|\beta| = 0$. For the convenience of the reader we give a short proof of claim iii) for arbitrary $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \tilde{m}$, now. Due to the boundedness of $\left\{ \varepsilon^{-t} \langle \xi \rangle^{-t} (1 - \phi(\varepsilon\xi)) : \varepsilon \in (0, 1] \right\} \subseteq S^0_{1,0}\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\right)$ and due to $\frac{\xi^\beta}{\langle \xi \rangle^{|\beta|}} \in S^0_{1,0}\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\right)$ we get the boundedness of

$$\left\{ \varepsilon^{-t} \xi^\beta \langle \xi \rangle^{-t} (1 - \phi(\varepsilon\xi)) : \varepsilon \in (0, 1] \right\} \subseteq S^{|\beta|}_{1,0}\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\right).$$

Since $\langle D_x \rangle^{-t}$ and D_x^β commute, we obtain claim iii) in the general case. □

Definition 3.4. Let $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$, $M \in \mathbb{N}_0 \cup \{\infty\}$, $m \in \mathbb{R}$ and $0 \le \delta \le \rho \le 1$. For $\gamma \in (\delta, 1)$ we set $\varepsilon_j := 2^{-j\gamma}$. For each $a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we define

- $a^\sharp(x, \xi) := \sum_{j=0}^\infty J_{\varepsilon_j} a(x, \xi) \psi_j(\xi)$ for all $x, \xi \in \mathbb{R}^n$,
- $a^b(x, \xi) := a(x, \xi) - a^\sharp(x, \xi)$ for all $x, \xi \in \mathbb{R}^n$.

Our aim is to verify useful properties of the functions $a^\sharp$ and a^b needed later on. To this end two new symbol-classes are needed, which we define, now.

Definition 3.5. Let $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$, $m \in \mathbb{R}$, $0 \le \delta, \rho \le 1$ and $M \in \mathbb{N}_0 \cup \{\infty\}$. Then $a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ *belongs to the symbol-class* $C^{\tilde{m},\tau} \dot{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, if for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \le M$ and $|\beta| \le \tilde{m}$ we have

$$\left| \partial_\xi^\alpha D_x^\beta a(x, \xi) \right| \le C_{\alpha,\beta}(x) \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|} \qquad \text{for all } x, \xi \in \mathbb{R}^n,$$

where $C_{\alpha,\beta}(x)$ is a bounded function, which converges to zero, as $|x| \to \infty$.

Moreover, $a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ belongs to the symbol-class $C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, if for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \tilde{m}$ and $|\beta| \neq 0$ we have

$$D_x^\beta a(x, \xi) \in C^{\tilde{m}-|\beta|,\tau} \dot{S}^{m+\delta|\beta|}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M).$$

We call the elements of $C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ *slowly varying symbols*. Moreover, $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathscr{L}\left(\mathbb{C}^N\right)$ is an element of the symbol-class $C^{\tilde{m},\tau} \dot{S}^m_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\left(\mathbb{C}^N\right)\right)$ respectively $C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\left(\mathbb{C}^N\right)\right)$, $m \in \mathbb{R}$, $N \in \mathbb{N}$, if and only if $a_{j,k} \in C^{\tilde{m},\tau} \dot{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ respectively $a_{j,k} \in C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for all $j, k = 1, \dots, N$, where we identify $A \in \mathscr{L}\left(\mathbb{C}^N\right)$ with a matrix $\left(a_{j,k}\right)_{j,k=1}^N \in \mathbb{C}^{N \times N}$ in the standard way.

The properties of the functions $a^\sharp$ and a^b are summarized in the next three lemmas:

Lemma 3.6. *Let* $0 \le \delta < \rho \le 1$, $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$, $M \in \mathbb{N} \cup \{\infty\}$, $m \in \mathbb{R}$ *and* $a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. *Moreover let* $\gamma \in (\delta, \rho)$. *Then we have for* $\tilde{\varepsilon} \in (0, (\gamma - \delta)\tau)$:

i) $D^\beta_x a^b(x,\xi) \in C^{\tilde{m}-|\beta|,\tau} S^{m-(\gamma-\delta)(\tilde{m}+\tau)+\gamma|\beta|}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; M)$ *for all* $\beta \in \mathbb{N}^n_0$ *with* $|\beta| \le \tilde{m}$,

ii) $a^b(x,\xi) \in C^{\tilde{m},\tau} \tilde{S}^{m-(\gamma-\delta)(\tilde{m}+\tau)+\tilde{\varepsilon}}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; M)$ *if* $a \in C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$,

iii) $a^b(x,\xi) \in C^{\tilde{m},\tau} \dot{S}^{m-(\gamma-\delta)(\tilde{m}+\tau)+\tilde{\varepsilon}}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; M)$ *if* $a \in C^{\tilde{m},\tau} \dot{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$.

Proof. We begin with the proof of i). We choose an arbitrary $\xi \in \mathbb{R}^n$ and set $N := \{ j \in \mathbb{N}_0 : \xi \in \operatorname{supp} \psi_j \}$. Then $\sharp N \le 5$. Using $a^\sharp(.,\xi) = \sum_{j \in N} J_{\varepsilon_j} a(.,\xi)\psi_j(\xi)$ and the Leibniz rule yields for all $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \le M$ and $|\beta| \le \tilde{m}$

$$\left|\partial^\alpha_\xi D^\beta_x a^b(x,\xi)\right| = \left|\partial^\alpha_\xi D^\beta_x \sum_{j=0}^\infty \left(1 - J_{\varepsilon_j}\right)\left(a(x,\xi)\psi_j(\xi)\right)\right|$$
$$\le \sum_{j\in N} \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha \left\| \left(1 - J_{\varepsilon_j}\right)\left(\partial^{\alpha_1}_\xi D^\beta_x a(x,\xi) \partial^{\alpha_2}_\xi \psi_j(\xi)\right)\right\|_{L^\infty(\mathbb{R}^n_x)}.$$

An application of Lemma 3.3 iv), (3.2) and (3.3) to the previous estimate provides:

$$\left|\partial^\alpha_\xi D^\beta_x a^b(x,\xi)\right| \le \sum_{j\in N} \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha \varepsilon_j^{\tilde{m}+\tau-|\beta|} \left\|\left(\partial^{\alpha_1}_\xi D^\beta_x a(x,\xi)\partial^{\alpha_2}_\xi \psi_j(\xi)\right)\right\|_{C^{\tilde{m}+\tau-|\beta|}(\mathbb{R}^n_x)}$$
$$\le \sum_{j\in N} \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha \langle\xi\rangle^{-\gamma(\tilde{m}+\tau-|\beta|)} \left|\partial^{\alpha_2}_\xi \psi_j(\xi)\right| \left\|\partial^{\alpha_1}_\xi D^\beta_x a(x,\xi)\right\|_{C^{\tilde{m}+\tau-|\beta|}(\mathbb{R}^n_x)}$$
$$\le C_{\alpha,\tilde{m},\tau} \langle\xi\rangle^{m-(\gamma-\delta)(\tilde{m}+\tau)+\gamma|\beta|-\rho|\alpha|} \qquad \text{for all } x,\xi \in \mathbb{R}^n. \tag{3.5}$$

Similarly we get by means of (3.4), the Leibniz rule, Lemma 3.3 iii) and (3.3) for all $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \le M$ and $|\beta| \le \tilde{m}$:

$$\left\|\partial^\alpha_\xi D^\beta_x a^b(.,\xi)\right\|_{C^{\tilde{m}-|\beta|,\tau}(\mathbb{R}^n)} \le C_{\alpha,\beta} \langle\xi\rangle^{m-(\gamma-\delta)(\tilde{m}+\tau)+\gamma|\beta|-\rho|\alpha|+\gamma(\tilde{m}-|\beta|+\tau)} \tag{3.6}$$

for all $\xi \in \mathbb{R}^n$. On account of (3.6) and (3.5) claim i) holds.

Our next goal is show ii) and iii). In order to prove the claim, we assume $a \in C^{\tilde{m},\tau} \dot{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ or $a \in C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Additionally we fix some arbitrary $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \le M$, $|\beta| \le \tilde{m}$ and $|\beta| \ne 0$ if $a \in C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. We choose an arbitrary $\varepsilon > 0$. As before we fix an arbitrary $\xi \in \mathbb{R}^n$ and set

$$N := \{ j \in \mathbb{N}_0 : \xi \in \operatorname{supp} \psi_j \}. \tag{13}$$

Moreover we define for all $j \in \mathbb{N}_0$ the functions $\varphi_{\varepsilon_j}, g_{\varepsilon_j}, g : \mathbb{R}^n \to \mathbb{C}$ via

- $\varphi_{\varepsilon_j} := \delta_0 - \mathscr{F}^{-1}_{\xi\to x}\left[\phi\left(\varepsilon_j \xi\right)\right]$ in $\mathcal{S}'(\mathbb{R}^n)$,
- $g_{\varepsilon_j}(x) := \mathscr{F}^{-1}_{\xi\to x}\left[\phi\left(\varepsilon_j \xi\right)\right](x)$ for all $x \in \mathbb{R}^n$,
- $g(x) := \mathscr{F}^{-1}_{\xi\to x}\left[\phi(\xi)\right](x)$ for all $x \in \mathbb{R}^n$.

By means of integration by parts and the Theorem of Fubini, we obtain for each $j \in \mathbb{N}$

$$\left[1 - \phi\left(\varepsilon_j D_x\right)\right] f = \varphi_{\varepsilon_j} * f(x) \qquad \text{for all } f \in C^0_b(\mathbb{R}^n). \tag{3.7}$$

Since we can change the order of the two operators D^β_x and $\left(1 - J_{\varepsilon_j}\right)$ an straight forward calculation yields if we use $a^\sharp(.,\xi) = \sum_{j\in N} J_{\varepsilon_j} a(.,\xi)\psi_j(\xi)$ and (3.7):

$$\left|\partial^\alpha_\xi D^\beta_x a^b(x,\xi)\right| = \left|\varphi_{\varepsilon_j} * \left\{\sum_{j\in N} \partial^\alpha_\xi \left[D^\beta_x a(.,\xi)\psi_j(\xi)\right]\right\}(x)\right|. \tag{3.8}$$

Our task is to use the previous equality in order to show for $\tilde{\varepsilon} \in (0, (\gamma - \delta)\tau)$:

$$\left|\partial_\xi^\alpha D_x^\beta a^b(x,\xi)\right| \leq C_{\alpha,\beta}(x)\langle\xi\rangle^{m+\delta|\beta|-(\gamma-\delta)(\tilde{m}-|\beta|+\tau)+\tilde{\varepsilon}-\rho|\alpha|} \xrightarrow{|x|\to\infty} 0. \tag{3.9}$$

Then a combination of (3.6), (3.5) and (3.9) yields claim *ii*) and *iii*). It remains to verify (3.9). The properties of the Fourier transform imply $g_{\varepsilon_j}, g \in \mathcal{S}(\mathbb{R}^n)$ for all $j \in \mathbb{N}_0$. Consequently $\langle y\rangle^{n+1} g_{\varepsilon_j}(y) \in \mathcal{S}(\mathbb{R}^n_y)$ for all $j \in N$. On account of the choice of a we get using (3.3):

$$\sum_{j\in N}\left|\partial_\xi^\alpha\left\{D_x^\beta a(x,\xi)\psi_j(\xi)\right\}\right| \leq A_1\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{3.10}$$

where A_1 is independent of $x, \xi \in \mathbb{R}^n$. Due to $\langle y\rangle^{n+1} g_{\varepsilon_j}(y) \in \mathcal{S}(\mathbb{R}^n_y)$ for all $j \in N$ we can choose an $R > 1$ such that for $A_2 := \int_{\mathbb{R}^n} \langle y\rangle^{-n-1}\,dy$ we have

$$\left|\langle y\rangle^{n+1} g_{\varepsilon_j}(y)\right| < \frac{\varepsilon}{2A_1A_2} \qquad \text{for all } y \in \mathbb{R}^n\backslash\overline{B_{R-1}(0)} \text{ and } j \in N. \tag{3.11}$$

In addition we choose an $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) \in [0,1]$, $\eta(x) = 1$ if $|x| \leq R-1$ and $\eta(x) = 0$ if $|x| \geq R$. Then we obtain for all $x \in \mathbb{R}^n$ by means of Lemma 3.3 iv), (3.10) and (3.11):

$$\begin{aligned}
&\left\|\left[\varphi_{\varepsilon_j}(1-\eta) * \sum_{j\in N}\partial_\xi^\alpha\left\{D_x^\beta a(.,\xi)\psi_j(\xi)\right\}\right](x)\right| \\
&\leq \int_{\mathbb{R}^n\backslash\overline{B_{R-1}(0)}} \left|\varphi_{\varepsilon_j}(y)\right||(1-\eta)(y)| \cdot \left\|\sum_{j\in N}\partial_\xi^\alpha\left\{D_x^\beta a(.,\xi)\psi_j(\xi)\right\}(x-y)\right\|_{L^\infty(\mathbb{R}^n_x)} dy \\
&\leq \frac{\varepsilon}{2}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}.
\end{aligned} \tag{3.12}$$

On account of the properties of the Fourier transform and due to the definition of φ_{ε_j} we get using $g \in \mathcal{S}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n\backslash\overline{B_{R-1}(0)}} \left|\varphi_{\varepsilon_j}\right| dy \leq \int_{\mathbb{R}^n} \varepsilon_j^{-n}\left|g\left(\frac{y}{\varepsilon_j}\right)\right| dy \leq \max\left\{1, \int_{\mathbb{R}^n} |g(z)|\,dz\right\} =: B_1 < \infty, \tag{3.13}$$

where B_1 is independent of $j \in \mathbb{N}$. The choice of the symbol a and the multi-index β gives us the existence of an $\tilde{R} > 0$ such that for all $|x| \geq \tilde{R} + R$ we have

$$\left|\sum_{j\in N}\partial_\xi^\alpha\left\{D_x^\beta a(.,\xi)\psi_j(\xi)\right\}(x)\right| \leq \frac{\varepsilon}{4B_1}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{3.14}$$

Using (3.8) we obtain for all $x \in \mathbb{R}^n$ with $|x| \geq \tilde{R} + R$:

$$\begin{aligned}
\left|\partial_\xi^\alpha D_x^\beta a^b(x,\xi)\right| \leq &\left|\int_{\mathbb{R}^n} \varepsilon_j^{-n} g\left(\frac{y}{\varepsilon_j}\right)\eta(y)\sum_{j\in N}\partial_\xi^\alpha\left[D_x^\beta a(x-y,\xi)\psi_j(\xi)\right]dy\right| \\
&+\left|\sum_{j\in N}\partial_\xi^\alpha\left[D_x^\beta a(x,\xi)\psi_j(\xi)\right]\right| + \left|\int_{\mathbb{R}^n}\varphi_{\varepsilon_j}(y)[1-\eta](y)\sum_{j\in N}\partial_\xi^\alpha\left[D_x^\beta a(x-y,\xi)\psi_j(\xi)\right]dy\right|.
\end{aligned}$$

Now we use (3.12) in order to estimate the third summand of the previous inequality and (3.14) to estimate the second summand of the previous inequality. The integrand of the first summand is always 0 if $|y| \geq R$. Hence we can estimate the first summand of the previous inequality by means of (3.14) and (3.13). Then we get $\left|\partial_\xi^\alpha D_x^\beta a^b(x,\xi)\right| \leq \varepsilon\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$ for all $x \in \mathbb{R}^n$ with

$|x| \geq \tilde{R} + R$. Hence

$$\left|\partial_\xi^\alpha D_x^\beta a^b(x,\xi)\right|\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|} \leq C_{\alpha,\beta}(x) \xrightarrow{|x|\to\infty} 0. \tag{3.15}$$

Now let $\tilde{\varepsilon}$ be as in the assumptions. Setting $\theta := \frac{(\gamma-\delta)(\tilde{m}-|\beta|+\tau)-\tilde{\varepsilon}}{(\gamma-\delta)(\tilde{m}-|\beta|+\tau)}$ we get by means of interpolation with (3.5) and (3.15), that estimate (3.9) holds:

$$\left|\partial_\xi^\alpha D_x^\beta a^b(x,\xi)\right|\langle\xi\rangle^{-m+(\gamma-\delta)(\tilde{m}-|\beta|+\tau)-\tilde{\varepsilon}+\rho|\alpha|-\delta|\beta|} \leq C_{\alpha,\beta}(x)^{1-\theta} C^\theta_{\alpha,\tilde{m},\tau} \xrightarrow{|x|\to\infty} 0.$$

Hence the lemma is proved. □

Lemma 3.7. *Let $0 \leq \delta < \rho \leq 1$, $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$, $M \in \mathbb{N} \cup \{\infty\}$, $m \in \mathbb{R}$ and $a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Moreover let $\gamma \in (\delta, \rho)$. Then we have for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$:*

i) $D_x^\beta a^\sharp(x,\xi) \in S^{m+\delta|\beta|}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; M\big)$,

ii) if $a \in C^{\tilde{m},\tau} \dot{S}^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M\big)$ or if $|\beta| \neq 0$ and $a \in C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M\big)$ then $D_x^\beta a^\sharp(x,\xi) \in \dot{S}^{m+\delta|\beta|}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; M\big)$.

Proof. Note that, because of $\left\|\mathscr{F}^{-1}(\phi(\varepsilon.))\right\|_{L^1(\mathbb{R}^n)} = \left\|\mathscr{F}^{-1}(\phi)\right\|_{L^1(\mathbb{R}^n)} =: C$,

$$\|\phi(\varepsilon D_x)\|_{\mathscr{L}(L^\infty(\mathbb{R}^n))} = \sup_{\|f\|_\infty \leq 1} \left\|\mathscr{F}^{-1}(\phi(\varepsilon.)) * f\right\|_\infty \leq C \qquad \text{for all } \varepsilon \in (0,1].$$

Now let $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$. We show, that for all $\tilde{\beta}, \alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq M$

$$\left\|D_x^{\tilde{\beta}} \partial_\xi^\alpha D_x^\beta a^\sharp(.,\xi)\right\|_\infty \leq C_{\alpha,\tilde{\beta},\beta}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|+\gamma|\tilde{\beta}|} \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{3.16}$$

This implies claim i). First of all we verify (3.16) for $\tilde{\beta} \in \mathbb{N}_0^n$ with $|\tilde{\beta}| \leq \tilde{m} - |\beta|$. To this end we choose an arbitrary $\xi \in \mathbb{R}^n$ with $N := \{j \in \mathbb{N}_0 : \xi \in \operatorname{supp} \psi_j\}$. Then $\sharp N \leq 5$. Using $a^\sharp(.,\xi) = \sum_{j\in N} J_{\varepsilon_j} a(.,\xi)\psi_j(\xi)$, the Leibniz rule, (3.3) and Lemma 2.2 yields for $\theta := \frac{|\tilde{\beta}|}{\tilde{m}+\tau-|\beta|}$

$$\begin{aligned}
\left\|D_x^{\tilde{\beta}} \partial_\xi^\alpha D_x^\beta a^\sharp(.,\xi)\right\|_\infty &\leq C_\alpha \sum_{j\in N} \sum_{\alpha_1+\alpha_2=\alpha} \langle\xi\rangle^{-\rho|\alpha_2|} \left\|\partial_\xi^{\alpha_1} D_x^\beta a(.,\xi)\right\|_{C_b^{|\tilde{\beta}|}(\mathbb{R}^n)} \\
&\leq C_\alpha \sum_{\alpha_1+\alpha_2=\alpha} \langle\xi\rangle^{-\rho|\alpha_2|} \left\|\partial_\xi^{\alpha_1} D_x^\beta a(.,\xi)\right\|^{1-\theta}_{C_b^0(\mathbb{R}^n)} \left\|\partial_\xi^{\alpha_1} D_x^\beta a(.,\xi)\right\|^\theta_{C^{\tilde{m}-|\beta|,\tau}(\mathbb{R}^n)} \\
&\leq C_{\alpha,\tilde{\beta},\beta}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|+\gamma|\tilde{\beta}|},
\end{aligned} \tag{3.17}$$

where $C_{\alpha,\tilde{\beta},\beta}$ is independent of $\xi \in \mathbb{R}^n$. Now let $\tilde{\beta} \in \mathbb{N}_0^n$ with $|\tilde{\beta}| + |\beta| \geq \tilde{m}$. Using $a^\sharp(.,\xi) = \sum_{j\in N} J_{\varepsilon_j} a(.,\xi)\psi_j(\xi)$, the Leibniz rule and (3.3) again, we obtain

$$\left\|D_x^{\tilde{\beta}} \partial_\xi^\alpha D_x^\beta a^\sharp(.,\xi)\right\|_\infty \leq C_\alpha \sum_{j\in N} \sum_{\alpha_1+\alpha_2=\alpha} \langle\xi\rangle^{-\rho|\alpha_2|} \left\|D_x^{\tilde{\beta}} J_{\varepsilon_j} \partial_\xi^{\alpha_1} D_x^\beta a(.,\xi)\right\|_\infty.$$

Now we can prove (3.16) by means of the previous inequality since $\partial_\xi^{\alpha_1} D_x^\beta a(.,\xi) \in C^{\tilde{m}-|\beta|,\tau}(\mathbb{R}^n)$ using Lemma 3.3 ii) and (3.2). It remains to prove claim ii). We again assume $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$. Moreover let $a \in C^{\tilde{m},\tau} \dot{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ or $|\beta| \neq 0$ and $a \in C^{\tilde{m},\tau} \tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$.

Similarly to the proof of (3.15) we will now show for $\alpha, \tilde{\beta} \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $|\tilde{\beta}| \leq \tilde{m} - |\beta|$:

$$\left|D_x^{\tilde{\beta}} \partial_\xi^\alpha D_x^\beta a^\sharp(.,\xi)\right| \leq C_{\alpha,\beta,\tilde{\beta}}(x)\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|+\gamma|\tilde{\beta}|} \qquad \text{for all } x, \xi \in \mathbb{R}^n. \tag{3.18}$$

Here $C_{\alpha,\beta,\tilde{\beta}}(x)$ is bounded and $C_{\alpha,\beta,\tilde{\beta}}(x) \xrightarrow{|x|\to\infty} 0$. In order to prove (3.18) for $\alpha, \tilde{\beta} \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $|\tilde{\beta}| + |\beta| \geq \tilde{m}$ we choose an arbitrary but fixed $\xi \in \mathbb{R}^n$ and define N as before. Additionally let $\varepsilon > 0$ be arbitrary. Since

$a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we get by means of the Leibniz rule and by (3.3) the existence of a constant $A_1 > 0$ with

$$\sum_{j\in N} \left|\partial_\xi^\alpha \left\{ D_x^\beta a(x,\xi)\psi_j(\xi)\right\}\right| \le A_1 \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{3.19}$$

Defining $g(\xi) := \xi^{\tilde{\beta}}\phi(\xi)$ for all $\xi \in \mathbb{R}^n$ we obtain for all $j \in \mathbb{N}$ and $f \in C_b^0(\mathbb{R}^n)$ due to the Theorem of Fubini:

$$\varepsilon_j^{|\tilde{\beta}|} D_x^{\tilde{\beta}} J_{\varepsilon_j}(D_x) f(x) = \int_{\mathbb{R}^n} \mathscr{F}^{-1}_{\xi\to x}\left[g(\varepsilon_j \xi)\right](x-y) f(y)\, dy. \tag{3.20}$$

Since $\phi(\varepsilon_j\xi) \in \mathcal{S}(\mathbb{R}^n_\xi)$, there is an $R > 1$ such that for all $|y| \ge R-1$

$$\left|\mathscr{F}^{-1}_{\xi\to x}\left[g(\varepsilon_j\xi)\right](y)\langle y\rangle^{n+1}\right| < \frac{\varepsilon}{2A_1A_2} \qquad \text{for all } j \in N, \tag{3.21}$$

where $A_2 := \int \langle y\rangle^{-n-1}\, dy$. Moreover we get on account of the properties of the Fourier transformation, change of variable and due to $g \in \mathcal{S}(\mathbb{R}^n)$:

$$B_3 := \int_{\mathbb{R}^n} \left|\mathscr{F}^{-1}_{\xi\to x}\left[g(\varepsilon_j\xi)\right](y)\right| dy = \int_{\mathbb{R}^n} \left|\mathscr{F}^{-1}[g](z)\right| dz < \infty. \tag{3.22}$$

The choice of the symbol a and of the multi-index β gives us the existence of an $\tilde{R} > 0$ such that for all $|x| \ge \tilde{R} + R - 1$ and for all $y \in \overline{B_{R-1}(0)}$ we have

$$\left|\sum_{j\in N} \partial_\xi^\alpha \left\{ D_x^\beta a(x-y,\xi)\psi(\xi)\right\}\right| \le \frac{\varepsilon}{2B_3}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \qquad \text{for all } |\beta| \ne 0. \tag{3.23}$$

Now let $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta(x) \in [0,1]$ for all $x \in \mathbb{R}^n$, $\eta(x) = 0$ for all $|x| \ge R$ and $\eta(x) = 1$ for all $|x| \le R-1$. By means of (3.19) and (3.21) we have

$$\begin{aligned} B_1 &:= \int_{\mathbb{R}^n\setminus \overline{B_{R-1}(0)}} \left|\mathscr{F}^{-1}_{\xi\to x}\left[g(\varepsilon_j\xi)\right](y)\right| |(1-\eta)(y)| \left|\sum_{j\in N} \partial_\xi^\alpha \left[D_x^\beta a(x-y,\xi)\psi_j(\xi)\right]\right| dy \\ &\le \frac{\varepsilon}{2}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \end{aligned} \tag{3.24}$$

Additionally a combination of (3.22) and (3.23) yields

$$\begin{aligned} B_2 &:= \int_{B_R(0)} \left|\mathscr{F}^{-1}_{\xi\to x}\left[g(\varepsilon_j\xi)\right](y)\right| |\eta(y)| \left|\sum_{j\in N} \partial_\xi^\alpha \left[D_x^\beta a(x-y,\xi)\psi_j(\xi)\right]\right| dy \\ &\le \frac{\varepsilon}{2}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \end{aligned} \tag{3.25}$$

Using $a^\sharp(.,\xi) = \sum_{j\in N} J_{\varepsilon_j} a(.,\xi)\psi_j(\xi)$, (3.20) and the definition of ε_j first and (3.24), (3.25) and (3.2) afterwards, we obtain for all $|x| \ge \tilde{R} + R - 1$:

$$\begin{aligned} \left|D_x^{\tilde{\beta}}\partial_\xi^\alpha D_x^\beta a^\sharp(x,\xi)\right| &= \varepsilon_j^{-|\tilde{\beta}|}\left|\varepsilon_j^{|\tilde{\beta}|} D_x^{\tilde{\beta}} J_{\varepsilon_j}\left\{\sum_{j\in N}\partial_\xi^\alpha\left[D_x^\beta a(x,\xi)\psi_j(\xi)\right]\right\}\right| \le 2^{j\gamma|\tilde{\beta}|}(B_1+B_2) \\ &\le 2^{j\gamma|\tilde{\beta}|}\varepsilon C\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|} \le \varepsilon C\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|+\gamma|\tilde{\beta}|}. \end{aligned}$$

Hence (3.18) also holds for $\alpha, \tilde{\beta} \in \mathbb{N}_0^n$ with $|\alpha| \le M$ and $|\tilde{\beta}| + |\beta| \ge \tilde{m}$. This provides ii). □

Lemma 3.8. *Let $0 \leq \delta < \rho \leq 1$, $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$, $M \in \mathbb{N} \cup \{\infty\}$, $m \in \mathbb{R}$ and $a \in C^{\tilde{m},\tau}\tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ such that*

$$a(x,\xi) \xrightarrow{|x|\to\infty} a(\infty,\xi) \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Moreover we set $b(x,\xi) := a(x,\xi) - a(\infty,\xi)$ for all $x,\xi \in \mathbb{R}^n$. Additionally we define $a^\sharp, a^b, a^\sharp(\infty,.)$ and $a^b(\infty,.)$ as in Definition 3.4. Then we have for $\gamma \in (\delta,\rho)$ and $\tilde{\varepsilon} \in (0,(\gamma-\delta)\tau)$:

i) $a^\sharp(\infty,\xi) = a(\infty,\xi) \in S^m_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; 0\big)$,

ii) $a^b(\infty,\xi) = 0$ *for all* $\xi \in \mathbb{R}^n$,

iii) $a^b(x,\xi) \in C^{\tilde{m},\tau}\tilde{S}^{m-(\gamma-\delta)(\tilde{m}+\tau)+\tilde{\varepsilon}}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap C^{\tilde{m},\tau}\dot{S}^{m-(\gamma-\delta)(\tilde{m}+\tau)+\tilde{\varepsilon}}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; 0\big)$,

iv) $a^\sharp(x,\xi) = a(\infty,\xi) + b^\sharp(x,\xi)$ *for all* $x,\xi \in \mathbb{R}^n$.

Proof. First of all we verify claim i). Since $a \in C^{\tilde{m},\tau}\tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we have

$$\|a(x,\xi)\langle\xi\rangle^{-m}\|_{C^{0,\tau}_b\left(\mathbb{R}^n_\xi\right)} \leq \|a(x,\xi)\langle\xi\rangle^{-m}\|_{C^1_b\left(\mathbb{R}^n_\xi\right)} \leq C \qquad \text{for all } x \in \mathbb{R}^n \text{ and}$$

$$|a(x,\xi)\langle\xi\rangle^{-m}| \leq C \qquad \text{for all } x,\xi \in \mathbb{R}^n. \tag{3.26}$$

Hence the definition of $C^{0,\tau}(\mathbb{R}^n)$ provides

$$|\langle\xi_1\rangle^{-m}a(x,\xi_1) - \langle\xi_2\rangle^{-m}a(x,\xi_2)| \leq C|\xi_1-\xi_2|^\tau \xrightarrow{\xi_1\to\xi_2} 0, \tag{3.27}$$

where C is independent of $x \in \mathbb{R}^n$. Taking $|x| \to \infty$ on both sides and using $\langle\xi\rangle^{-m} \in C^\infty(\mathbb{R}^n)$ yields $a(\infty,\xi) \in C^0\big(\mathbb{R}^n_\xi\big)$. Taking $|x| \to \infty$ on both sides of (3.26) provides

$$|a(\infty,\xi)| \leq C\langle\xi\rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Together with (3.27) we therefore get

$$a(\infty,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; 0).$$

By means of Remark 2.8 we can show for all $\xi \in \mathbb{R}^n$

$$J_\varepsilon a(\infty,\xi) = a(\infty,\xi) \cdot \text{Os -}\iint e^{-iz\cdot\eta}\phi(\varepsilon\eta)\,dz\,đ\eta = a(\infty,\xi).$$

Hence we obtain for all $\xi \in \mathbb{R}^n$

$$a^\sharp(\infty,\xi) = a(\infty,\xi) \qquad \text{and} \qquad a^b(\infty,\xi) = a(\infty,\xi) - a^\sharp(\infty,\xi) = 0.$$

This provides i), ii) and iv). It remains to verify claim iii). On account of the definition of $a(\infty,\xi)$ and $a \in C^{\tilde{m},\tau}\tilde{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we have for all $\beta \in \mathbb{N}^n_0$ with $|\beta| \leq \tilde{m}$

$$\left|D^\beta_x b(x,\xi)\right| \leq C_\beta(x)\langle\xi\rangle^{m+\delta|\beta|} \qquad \text{for all } \xi \in \mathbb{R}^n, \tag{3.28}$$

where $C_\beta(x) \to 0$ if $|x| \to \infty$. Moreover

$$\|a(\infty,\xi)\|_{C^{\tilde{m},\tau}(\mathbb{R}^n_x)} \leq |a(\infty,\xi)| = \left|\lim_{|x|\to\infty} a(x,\xi)\right| \leq C\langle\xi\rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n,$$

we get $\|b(.,\xi)\|_{C^{\tilde{m},\tau}(\mathbb{R}^n)} \leq \langle\xi\rangle^{m+\delta\cdot(\tilde{m}+\tau)}$. Together with (3.28) this yields

$$b(x,\xi) \in C^{\tilde{m},\tau}\dot{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; 0).$$

Consequently Lemma 3.6 and $a^b(x,\xi) = b^b(x,\xi)$ provides claim iii). □

3.2 | Symbol reduction

In this subsection we prove a formula representing an operator with a non-smooth double symbol as an operator with a non-smooth single symbol. Non-smooth double symbols are defined in the following way:

Definition 3.9. Let $\tilde{m} \in \mathbb{N}_0$, $0 < \tau < 1$, $m_1, m_2 \in \mathbb{R}$, $0 \leq \delta, \rho \leq 1$ and $M_1, M_2 \in \mathbb{N}_0 \cup \{\infty\}$. Then a continuous function $a : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'} \to \mathbb{C}$ *belongs to the non-smooth double symbol-class* $C^{\tilde{m},\tau} S^{m_1,m_2}_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1, M_2\right)$ if

i) $\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} a \in C^{\tilde{m},\tau}\left(\mathbb{R}^n_x\right)$ and $\partial_x^\beta \partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} a \in C^0\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'}\right)$,

ii) $\left|\partial_x^\beta \partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} a(x,\xi,x',\xi')\right| \leq C_{\alpha,\beta,\beta',\alpha'}(x) \tilde{C}_{\alpha,\beta,\beta',\alpha'}(x') \langle \xi \rangle^{m_1 - \rho|\alpha| + \delta|\beta|} \langle \xi' \rangle^{m_2 - \rho|\alpha'|} \langle \xi; \xi' \rangle^{\delta|\beta'|}$,

iii) $\left\|\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} a(.,\xi,x',\xi')\right\|_{C^{\tilde{m},\tau}(\mathbb{R}^n)} \leq C_{\alpha,\beta',\alpha'} \langle \xi \rangle^{m_1 - \rho|\alpha| + \delta(\tilde{m}+\tau)} \langle \xi' \rangle^{m_2 - \rho|\alpha'|} \langle \xi; \xi' \rangle^{\delta|\beta'|}$,

for all $x, \xi, x', \xi' \in \mathbb{R}^n$ and arbitrary $\beta, \alpha, \beta', \alpha' \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$, $|\alpha| \leq M_1$ and $|\alpha'| \leq M_2$. Here the constants $C_{\alpha,\beta,\beta',\alpha'}(x)$, $C_{\alpha,\beta',\alpha'}$ and $\tilde{C}_{\alpha,\beta,\beta',\alpha'}(x')$ are bounded and independent of $\xi, x', \xi' \in \mathbb{R}^n$ respectively $\xi, x, \xi' \in \mathbb{R}^n$.

If we even have $C_{\alpha,\beta,\beta',\alpha'}(x) \xrightarrow{|x|\to\infty} 0$ for all $\beta, \alpha, \beta', \alpha' \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$, $|\alpha| \leq M_1$ and $|\alpha'| \leq M_2$, then a is an element of $C^{\tilde{m},\tau} \dot{S}^{m_1,m_2}_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1, M_2\right)$. If we have $\tilde{C}_{\alpha,\beta,\beta',\alpha'}(x') \xrightarrow{|x'|\to\infty} 0$ for all $\beta, \alpha, \beta', \alpha' \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$, $|\alpha| \leq M_1$ and $|\alpha'| \leq M_2$ instead, then a is an element of $C^{\tilde{m},\tau} \hat{S}^{m_1,m_2}_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1, M_2\right)$.

For each double symbol $a \in C^{\tilde{m},\tau} S^{m_1,m_2}_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1, M_2\right)$ we define the associated pseudodifferential operator P by

$$Pu(x) := \mathrm{Os}\text{-}\iiiint e^{-i(y\cdot\xi + y'\cdot\xi')} a(x,\xi,x+y,\xi') u(x+y+y')\, dy\, dy'\, đ\xi\, đ\xi'$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

In the smooth case, i.e. if $M_1, M_2 = \infty$, the symbol-reduction is well-known, cf. e.g. [11, Lemma 2.4]. For non-smooth double symbols of the symbol-class $C^{\tilde{m},\tau} S^{m,m'}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N, \infty)$ the symbol smoothing was proved in [10, Theorem 3.33] in the case $N = \infty$ and in [2, Section 4.2] in the case $(\rho,\delta) = (0,0)$. As an ingredient for the proof of the Fredholm property of non-smooth pseudodifferential operators, we need the symbol reduction in a more general setting.

Theorem 3.10. *Let* $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$ *and* $m_1, m_2 \in \mathbb{R}$. *Additionally we choose* $N_1, N_2 \in \mathbb{N}_0 \cup \{\infty\}$ *such that there is an* $l \in \mathbb{N}$ *with* $N_1 \geq l > n$. *Moreover, we define* $\tilde{N} := \min\{N_1 - (n+1), N_2\}$. *Furthermore, let*

$$\mathscr{B} \subseteq C^{\tilde{m},s} S^{m_1,m_2}_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N_1, N_2\right)$$

be bounded. If we define for each $a \in \mathscr{B}$ *and* $\theta \in [0,1]$ *the function* $a_L^\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *by*

$$a_L^\theta(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \theta\eta + \xi, x+y, \xi)\, dy\, đ\eta \qquad \text{for all } x, \xi \in \mathbb{R}^n,$$

we get with $m := m_1 + m_2$ *that* $a_L^\theta \in C^{\tilde{m},s} S^m_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N}\right)$ *for all* $a \in \mathscr{B}$ *and* $\theta \in [0,1]$ *and the existence of a constant* C_α, *independent of* $a \in \mathscr{B}$ *and* $\theta \in [0,1]$, *such that for all* $\alpha, \beta \in \mathbb{N}_0^n$ *with* $|\alpha| \leq \tilde{N}$ *and* $|\beta| \leq \tilde{m}$

$$\left\|\partial_\xi^\alpha a_L^\theta(.,\xi)\right\|_{C^{\tilde{m},s}(\mathbb{R}^n)} \leq C_\alpha \langle \xi \rangle^{m - \rho|\alpha| + \delta(\tilde{m}+s)} \qquad \text{for all } \xi \in \mathbb{R}^n \tag{3.29}$$

and

$$\left|\partial_\xi^\alpha \partial_x^\beta a_L^\theta(x,\xi)\right| \leq C_{\alpha,\beta}(x) \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|} \qquad \text{for all } \xi \in \mathbb{R}^n, \tag{3.30}$$

where $C_{\alpha,\beta}(x)$ *is bounded and independent of* $a \in \mathscr{B}$, $\xi \in \mathbb{R}^n$ *and* $\theta \in [0,1]$. *This implies the boundedness of* $\left\{a_L^\theta : a \in \mathscr{B}, \theta \in [0,1]\right\} \subseteq C^{\tilde{m},s} S^m_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N}\right)$. *If* $\mathscr{B}$ *is even a bounded set in* $C^{\tilde{m},s} \dot{S}^{m_1,m_2}_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N_1, N_2\right)$ *or in* $C^{\tilde{m},s} \hat{S}^{m_1,m_2}_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N_1, N_2\right)$, *then* $C_{\alpha,\beta}(x) \xrightarrow{|x|\to\infty} 0$.

We combine the ideas of the smooth symbol reduction in [11, Lemma 2.4] and that one in [2, Section 4.2] in order to get the boundedness of $\{a_L^\theta : a \in \mathscr{B}, \theta \in [0,1]\} \subseteq C^{\tilde{m},s} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N})$. To show $C_{\alpha,\beta}(x) \xrightarrow{|x|\to\infty} 0$ additionally some new arguments are needed. Unfortunately one looses some regularity with respect to the second variable of the order $n+1$ in the proof. The ability to treat the even and odd space dimensions in the same way is based on the next remark:

Remark 3.11. Let $l \in \mathbb{N}$ be arbitrary. Then

$$e^{iy\cdot\eta} = \Big\{ \big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{-(l+1)} \big(1 + \langle\xi\rangle^{2\delta}(-\Delta_\eta)\big)^l + \sum_{j=1}^n \big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{-(2l+1)/2} \frac{\langle\xi\rangle^\delta y_j}{\big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{1/2}} \big(1 + \langle\xi\rangle^{2\delta}(-\Delta_\eta)\big)^l \langle\xi\rangle^\delta D_{\eta_j} \Big\} e^{iy\cdot\eta}$$

and we have for all $l_0 \in \mathbb{N}$, $\gamma \in \mathbb{N}_0^n$

$$\left|\partial_y^\gamma \big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{-l_0}\right| \le C_{l_0,\gamma} \langle\xi\rangle^{\delta|\gamma|} \big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{-l_0} \qquad \text{for all } y,\xi \in \mathbb{R}^n. \tag{3.31}$$

We additionally have for all $\gamma \in \mathbb{N}_0^n$:

$$\left|\partial_y^\gamma \frac{\langle\xi\rangle^\delta y_j}{\big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{1/2}}\right| \le \langle\xi\rangle^{\delta|\gamma|}. \tag{3.32}$$

Definition 3.12. Let $l \in \mathbb{N}$ be arbitrary. Then we define

$$B^l(y, \Delta_\eta) := \big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{-l/2} \big(1 + \langle\xi\rangle^{2\delta}(-\Delta_\eta)\big)^{l/2}$$

if l is even, and

$$B^l(y, \Delta_\eta) := \big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{-l/2-1/2} \big(1 + \langle\xi\rangle^{2\delta}(-\Delta_\eta)\big)^{(l-1)/2} + \sum_{j=1}^n \big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{-l/2} \frac{\langle\xi\rangle^\delta y_j}{\big(1 + \langle\xi\rangle^{2\delta}|y|^2\big)^{1/2}} \big(1 + \langle\xi\rangle^{2\delta}(-\Delta_\eta)\big)^{(l-1)/2} \langle\xi\rangle^\delta D_{\eta_j}$$

else for all $y, \xi \in \mathbb{R}^n$.

In order to improve the symbol reduction, we need the next result:

Proposition 3.13. *Let $0 \le \delta \le \rho \le 1$ with $\delta \ne 1$, $0 < \tau < 1$, $\tilde{m} \in \mathbb{N}_0$ and $m_1, m_2 \in \mathbb{R}$. Additionally let $N_1, N_2 \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $n < l \le N_1$. Moreover, let $a \in C^{\tilde{m},\tau} S^{m_1,m_2}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N_1, N_2)$. Considering an $l_0 \in \mathbb{N}_0$ with $n < l_0 \le N_1$, we define $r^\theta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ for all $\theta \in [0,1]$ by*

$$r^\theta(x, \xi, y, \eta) := B^{l_0}(y, \Delta_\eta) a(x, \xi + \theta\eta, x + y, \xi)$$

for all $x, \xi, \eta, y \in \mathbb{R}^n$. Then we have $r^\theta(x, \xi, y, \eta) \in L^1(\mathbb{R}^n_y)$ for all $x, \xi, \eta \in \mathbb{R}^n$ and $\int e^{-iy\cdot\eta} r^\theta(x, \xi, y, \eta)\, dy \in L^1(\mathbb{R}^n_\eta)$ for all $x, \xi \in \mathbb{R}^n$. Moreover we obtain

$$Os\text{-}\iint e^{-iy\cdot\eta} r^\theta(x, \xi, y, \eta)\, dy\, đ\eta = \int \left[\int e^{-iy\cdot\eta} r^\theta(x, \xi, y, \eta)\, dy\right] đ\eta.$$

Proof. First of all we prove the claim for even l_0 and use $2l_0$ instead of l_0. Let $x, \xi \in \mathbb{R}^n$ be arbitrary. We define $m := m_1 + m_2$. For every $\tilde{\gamma} \in \mathbb{N}_0^n$ we get due to $a \in C^{\tilde{m},\tau} S^{m_1,m_2}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N_1, N_2)$, the Leibniz rule and $\langle\xi + \theta\eta; \xi\rangle \le \langle\xi\rangle\langle\eta\rangle$ for $\tilde{l} \in \mathbb{N}_0, \tilde{l} \le l_0$:

$$\left|\partial_y^{\tilde{\gamma}} \Big\{ \big[\langle\xi\rangle^{2\delta}(-\Delta_\eta)\big]^{\tilde{l}} \Big) a(x, \xi + \theta\eta, x + y, \xi) \Big\}\right| \le C_{\tilde{l},\tilde{\gamma}} \langle\eta\rangle^{|m_1| + \delta|\tilde{\gamma}|} \langle\xi\rangle^{m + \delta|\tilde{\gamma}| + 2\tilde{l}\delta} \tag{3.33}$$

for all $y, \eta \in \mathbb{R}^n$, where $C_{\tilde{l},\tilde{\gamma}}$ is independent of $x, y, \xi, \eta \in \mathbb{R}^n$, $\theta \in [0,1]$. Now the Leibniz rule provides for all $l \in \mathbb{N}_0$ by means of (3.33) and (3.31) the existence of a $C_l > 0$, independent of $x, y, \xi, \eta \in \mathbb{R}^n$, $\theta \in [0,1]$, such that

$$\begin{aligned}\left|\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l} r^\theta(x,\xi,y,\eta)\right| &\leq C_l\langle\eta\rangle^{-2l}\left(1+\langle\xi\rangle^{2\delta}|y|^2\right)^{-l_0}\langle\eta\rangle^{|m_1|+2l\delta}\langle\xi\rangle^{m+2l\delta+2l_0\delta}\\ &\leq C_l\langle y\rangle^{-2l_0}\langle\eta\rangle^{|m_1|-2l(1-\delta)}\langle\xi\rangle^{m+2l\delta+2l_0\delta}\end{aligned}$$

for all $\xi, \eta \in \mathbb{R}^n$.

Assuming an arbitrary $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$, we get for fixed $x, \eta, \xi \in \mathbb{R}^n$:

$$e^{-iy\cdot\eta}\chi(\varepsilon y)r^\theta(x,\xi,y,\eta) \xrightarrow{\varepsilon\to 0} e^{-iy\cdot\eta}r^\theta(x,\xi,y,\eta) \quad \text{pointwise for all } y \in \mathbb{R}^n. \tag{3.34}$$

Now let $0 < \varepsilon \leq 1$. Using the Leibniz rule and $\chi \in \mathcal{S}(\mathbb{R}^n) \subseteq C_b^\infty(\mathbb{R}^n)$ we have

$$\left|\langle\eta\rangle^{-2l'}\langle D_y\rangle^{2l'}\left[\chi(\varepsilon y)r^\theta(x,\xi,y,\eta)\right]\right| \leq C_l\langle y\rangle^{-2l_0}\langle\eta\rangle^{|m_1|-2l'(1-\delta)}\langle\xi\rangle^{m+2l'\delta+2l_0\delta}, \tag{3.35}$$

for all $l' \in \mathbb{N}_0$ uniformly in $x, \xi, \eta, y \in \mathbb{R}^n$ and in $0 < \varepsilon \leq 1$. Integration by parts yields for arbitrary $\ell \in \mathbb{N}_0$ with $|m_1| - 2\ell(1-\delta) < -n$:

$$\int e^{-iy\cdot\eta}\chi(\varepsilon y)r^\theta(x,\xi,y,\eta)\,dy = \int e^{-iy\cdot\eta}\langle\eta\rangle^{-2\ell}\langle D_y\rangle^{2\ell}\left[\chi(\varepsilon y)r^\theta(x,\xi,y,\eta)\right]dy. \tag{3.36}$$

Using $\chi \in \mathcal{S}(\mathbb{R}^n) \subseteq C_b^\infty(\mathbb{R}^n)$ and (3.36) first and (3.35) afterwards provides for fixed $x, \xi \in \mathbb{R}^n$:

$$\left|\chi(\varepsilon\eta)\int e^{-iy\cdot\eta}\chi(\varepsilon y)r^\theta(x,\xi,y,\eta)\,dy\right| \leq C_{l,m,\xi}\langle\eta\rangle^{|m_1|-2\ell(1-\delta)} \in L^1\left(\mathbb{R}^n_\eta\right). \tag{3.37}$$

Here the constant $C_{\ell,m,\xi}$ is independent of $\varepsilon \in (0,1]$ and $x \in \mathbb{R}^n$. Setting $l' = 0$ in (3.35) we obtain for each fixed $x, \xi, \eta \in \mathbb{R}^n$, that

$$\left\{y \mapsto \chi(\varepsilon y)r^\theta(x,\xi,y,\eta) : 0 < \varepsilon \leq 1\right\}$$

has a $L^1\left(\mathbb{R}^n_y\right)$-majorant. Together with (3.34) and (3.37) we have verified all assumptions of Lebesgue's theorem. An application of Lebesgue's theorem two times provides

$$\text{Os-}\iint e^{-iy\cdot\eta}r^\theta(x,\xi,y,\eta)\,dy\,đ\eta = \int\left[\int e^{-iy\cdot\eta}r^\theta(x,\xi,y,\eta)\,dy\right]đ\eta.$$

If l_0 is odd, we can prove the claim in the same way, using Remark 3.11. □

Proposition 3.14. *Let $0 < \delta < 1$, $m_1, m_2 \in \mathbb{R}$, $u \geq 0$ and $\theta \in [0,1]$. Additionally let X be a Banach space such that $X \subseteq C_b^0(\mathbb{R}^n)$. Considering $l_0 \in \mathbb{N}_0$ with $-l_0 < -n$, we choose a set $\mathscr{B}$ of functions $r : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that the next inequality holds for all $l \in \mathbb{N}_0$:*

$$\begin{aligned}\left|(-\Delta_y)^l r^\theta(x,\xi,y,\eta)\right| &\leq C_l(x)\tilde{C}_l(x+y)\left(1+\langle\xi\rangle^{2\delta}|y|^2\right)^{-l_0/2}\langle\xi+\theta\eta\rangle^{m_1}\langle\xi\rangle^{m_2}\langle\xi+\theta\eta;\xi\rangle^{2l\delta+u},\\ \left\|(-\Delta_y)^l r^\theta(.,\xi,y,\eta)\right\|_X &\leq \hat{C}_l\left(1+\langle\xi\rangle^{2\delta}|y|^2\right)^{-l_0/2}\langle\xi+\theta\eta\rangle^{m_1}\langle\xi\rangle^{m_2}\cdot\langle\xi+\theta\eta;\xi\rangle^{2l\delta+u}.\end{aligned}$$

Here the constants $C_l(x), \tilde{C}_l(x+y)$, $\hat{C}_l$ are bounded and independent of $\xi, \eta \in \mathbb{R}^n$, $\theta \in [0,1]$ and of $r \in \mathscr{B}$. $\hat{C}_l$ is also independent of x and y. If we denote the sets $\Omega_1 := \left\{\eta \in \mathbb{R}^n : |\eta| \leq \frac{1}{2}\langle\xi\rangle^\delta\right\}$, $\Omega_2 := \left\{\eta \in \mathbb{R}^n : \frac{1}{2}\langle\xi\rangle^\delta \leq |\eta| \leq \frac{1}{2}\langle\xi\rangle\right\}$ and $\Omega_3 := \left\{\eta \in \mathbb{R}^n : |\eta| \geq \frac{1}{2}\langle\xi\rangle\right\}$ first and define

$$I_i^\theta(x,\xi) := \int_{\Omega_i}\int_{\mathbb{R}^n} e^{-iy\cdot\eta}r^\theta(x,\xi,y,\eta)\,dy\,đ\eta \qquad \text{for } i \in \{1,2,3\}$$

for arbitrary $x, \xi \in \mathbb{R}^n$ afterwards, then there are constants $C(x)$, $\hat{C}$, bounded and independent of $\xi \in \mathbb{R}^n$, $\theta \in [0,1]$ and $r \in \mathscr{B}$, such that

$$\left|I_i^\theta(x,\xi)\right| \le C(x)\langle\xi\rangle^m, \quad \left\|I_i^\theta(\cdot,\xi)\right\|_X \le \hat{C}\langle\xi\rangle^m \qquad \text{for } i \in \{1,2,3\}, \tag{3.38}$$

where $m := m_1 + m_2 + u$. Here $\hat{C}$ is independent of x. If $C_l(x) \xrightarrow{|x|\to\infty} 0$ or $\tilde{C}_l(x+y) \xrightarrow{|x+y|\to\infty} 0$ for all $l \in \mathbb{N}$, then $C(x) \xrightarrow{|x|\to\infty} 0$.

Proof. First of all we prove the claim for even l_0 and use $2l_0$ instead of l_0. Let $\xi \in \mathbb{R}^n$. In the following we will for simplicity write $\|\cdot\|$ for $|.|$ or $\|.\|_X$.

The assumptions and $\langle\xi+\theta\eta;\xi\rangle \le \langle\xi\rangle\langle\eta\rangle$ give us the existence of bounded constants $C_l(x)$, $\tilde{C}_l(x+y)$, independent of $\xi,\eta \in \mathbb{R}^n$, $r \in \mathscr{B}$ and in the case $\|.\| = \|.\|_X$ also independent of x and y, such that

$$\left\|\left(-\Delta_y\right)^l r^\theta\left(.,\xi,y,\eta\right)\right\| \le C_l(x)\tilde{C}_l(x+y)\left(1+\langle\xi\rangle^{2\delta}|y|^2\right)^{-l_0}\langle\xi+\theta\eta\rangle^{m_1}\langle\xi\rangle^{m_2}\langle\xi+\theta\eta;\xi\rangle^{2l\delta+u} \tag{3.39}$$

$$\le C_l(x)\tilde{C}_l(x+y)\langle y\rangle^{-2l_0}\langle\xi\rangle^{m_1+m_2+2l\delta+u}\langle\eta\rangle^{|m_1|+2l\delta+u} \in L^1\left(\mathbb{R}^n_y\right) \tag{3.40}$$

for all $\xi,\eta \in \mathbb{R}^n, \theta \in [0,1], l \in \mathbb{N}_0$. For all $\eta \in \Omega_1 \cup \Omega_2$ and $m_1 \in \mathbb{R}$ the estimates $\langle\xi+\theta\eta\rangle^{m_1} \le C_{m_1}\langle\xi\rangle^{m_1}$ and $\langle\xi+\theta\eta;\xi\rangle^{2l\delta+u} \le C\langle\xi\rangle^{2l\delta+u}$ hold. Now let $m := m_1+m_2+u$. Then we can simplify (3.39) for all $\eta \in \Omega_1 \cup \Omega_2$ to

$$\left\|\left(-\Delta_y\right)^l r^\theta\left(.,\xi,y,\eta\right)\right\| \le C_l(x)\tilde{C}_l(x+y)\left(1+\langle\xi\rangle^{2\delta}|y|^2\right)^{-l_0}\langle\xi\rangle^{m+2l\delta} \tag{3.41}$$

for all $\xi, y \in \mathbb{R}^n, l \in \mathbb{N}_0$, where $C_l(x), \tilde{C}_l(x+y)$ are bounded and independent of $\theta \in [0,1]$, $\xi,\eta \in \mathbb{R}^n$, $r \in \mathscr{B}$ and in the case $\|.\| = \|.\|_X$ also independent of x and y. In order to estimate $\|I_1\|$, we also need the following calculation, which can be verified by means of the change of variables $\tilde{\eta} := \langle\xi\rangle^{-\delta}\eta$:

$$\int_{|\eta|\le 0.5\langle\xi\rangle^\delta} đ\eta = \langle\xi\rangle^{\delta n}\int_{|\tilde{\eta}|\le 0.5} đ\eta \le C_n\langle\xi\rangle^{\delta n}. \tag{3.42}$$

Thus a combination of (3.41) and (3.42) concludes together with a change of variable $w := \langle\xi\rangle^\delta y$:

$$\left\|I_1^\theta\right\| \le C_1(x)\langle\xi\rangle^{m-\delta n}\int_{\Omega_1}\int_{\mathbb{R}^n}\tilde{C}_0\left(x+\langle\xi\rangle^{-\delta}w\right)\left(1+|w|^2\right)^{-l_0}dw\,đ\eta \le C_1(x)\langle\xi\rangle^m,$$

where $C_1(x)$ is bounded and independent of $\xi \in \mathbb{R}^n$, $r \in \mathscr{B}$ and in the case that $\|.\| = \|.\|_X$ also independent of x and y. For the estimate of $\|I_2\|$ and $\|I_3\|$ we choose $l \in \mathbb{N}_0$ with $-2l < -n$. Together with the equation $e^{-iy\cdot\eta} = |\eta|^{-2l}\left(-\Delta_y\right)^l e^{-iy\cdot\eta}$ we obtain by integration by parts:

$$\int_{\mathbb{R}^n} e^{-iy\cdot\eta}r^\theta\left(x,\xi,y,\eta\right)dy = |\eta|^{-2l}\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\left(-\Delta_y\right)^l r^\theta\left(x,\xi,y,\eta\right)dy. \tag{3.43}$$

Additonally we have

$$\int_{|\eta|\ge 0.5\langle\xi\rangle^\delta}|\eta|^{-2l}\,đ\eta = C_n\left|\int_{0.5\langle\xi\rangle^\delta}^\infty r^{n-1-2l}\,dr\right| = C_{n,l}\langle\xi\rangle^{(-2l+n)\delta}. \tag{3.44}$$

If we utilize (3.43) and (3.41) first, and (3.44) afterwards, we obtain

$$\left\|I_2^\theta\right\| \le C_2(x)\langle\xi\rangle^{m+2l\delta-\delta n}\int_{\Omega_2}|\eta|^{-2l}\int_{\mathbb{R}^n}\tilde{C}_l\left(x+\langle\xi\rangle^{-\delta}w\right)\left(1+|w|^2\right)^{-l_0}dw\,đ\eta \le C_{2,l}(x)\langle\xi\rangle^m,$$

where $C_{2,l}(x)$ is bounded and independent of $\xi \in \mathbb{R}^n$, $\theta \in [0,1]$, $r \in \mathscr{B}$ and in the case $\|.\| = \|.\|_X$ also independent of x and y. It remains to estimate $\left\|I_3^\theta\right\|$. For each $\eta \in \Omega_3$, we have $\langle\xi+\theta\eta\rangle \le \langle\xi\rangle + |\theta\eta| \le 3|\eta|$ and $\langle\xi+\theta\eta;\xi\rangle \le \sqrt{13}|\eta|$. Denoting

$k_+ := \max\{0,k\}$ and $k_- := \min\{0,k\}$ this provides together with (3.39) the existence of some constants $C_l(x), \tilde{C}_l(x+y)$, bounded and independent of $\xi \in \mathbb{R}^n$, $\eta \in \Omega_3$, $\theta \in [0,1]$, $r \in \mathscr{B}$ and in the case $\|.\| = \|.\|_X$ also independent of x and y, such that

$$|\eta|^{-2l}\left\|\left(-\Delta_y\right)^l r^\theta\left(.,\xi,y,\eta\right)\right\| \leq C_l(x)\tilde{C}_l(x+y)\left(1+\langle\xi\rangle^{2\delta}|y|^2\right)^{-l_0}|\eta|^{(m_1)_+ + u - 2l(1-\delta)}\langle\xi\rangle^{m_2} \tag{3.45}$$

for all $\xi, y \in \mathbb{R}^n$ and $\eta \in \Omega_3$. Analog to the calculation of (3.44) we get

$$\int_{\Omega_3} |\eta|^{(m_1)_+ + u - 2l(1-\delta)}\, đ\eta \leq C\langle\xi\rangle^{(m_1)_+ - 2l(1-\delta)+n-m_1}\langle\xi\rangle^{m_1+u+\delta n} \leq C\langle\xi\rangle^{m_1+u+\delta n}, \tag{3.46}$$

if we choose an $l \in \mathbb{N}_0$ with $-(m_1)_- + u - 2l(1-\delta) \leq -n$. Finally a combination of (3.43), (3.45) and (3.46) concludes similarly to the estimates of $\|I_2^\theta\|$:

$$\left\|I_3^\theta\right\| \leq C_3(x)\langle\xi\rangle^m.$$

Here $C_3(x)$ is bounded and independent of $\xi \in \mathbb{R}^n$, $\theta \in [0,1]$, $r \in \mathscr{B}$ and in the case $\|.\| = \|.\|_X$ also independent of x and y. If $\|.\| = |.|$ and $C_l(x) \xrightarrow{|x|\to\infty} 0$ for all $l \in \mathbb{N}$, we get by verifying the proof, that $C(x) \xrightarrow{|x|\to\infty} 0$.

Now assume, that $\|.\| = |.|$ and that for all $l \in \mathbb{N}_0$ we have $\tilde{C}_l(x+y) \xrightarrow{|x+y|\to\infty} 0$ and $\tilde{C}_l(x+y) \leq B_l$ for all $x, y \in \mathbb{R}^n$. In order to verify that $C(x) \xrightarrow{|x|\to\infty} 0$ in estimate (3.38), we choose an arbitrary $l \in \mathbb{N}_0$ and $\varepsilon > 0$. Additionally let $\tilde{\varepsilon} > 0$ with $-l_0 < -l_0 + \tilde{\varepsilon} < -n$ be arbitrary but fixed. Defining $A := \int_{\mathbb{R}^n} \langle w\rangle^{-l_0+\tilde{\varepsilon}} dw$ we obtain due to $\langle w\rangle^{-\tilde{\varepsilon}} \in S\left(\mathbb{R}^n_w\right)$ the existence of a $R > 0$ such that

$$\langle w\rangle^{-\tilde{\varepsilon}} \leq \frac{\varepsilon}{2AB_l} \qquad \text{for all } w \in \mathbb{R}^n \backslash B_R(0). \tag{3.47}$$

Since $\tilde{C}_l(x+y) \xrightarrow{|x+y|\to\infty} 0$, there is a $\tilde{R} > 0$ such that

$$\tilde{C}_l(x+y) \leq \frac{\varepsilon}{2A} \qquad \text{for all } x, y \in \mathbb{R}^n \text{ with } |x+y| \geq \tilde{R}. \tag{3.48}$$

Using (3.47) and (3.48) we obtain for all $x \in \mathbb{R}^n$ with $|x| \geq \tilde{R} + R$:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \tilde{C}_l\left(x+\langle\xi\rangle^{-\delta}w\right)\langle w\rangle^{-l_0}\, dw \\
&= \int_{\mathbb{R}^n\backslash B_R(0)} \tilde{C}_l\left(x+\langle\xi\rangle^{-\delta}w\right)\langle w\rangle^{-\tilde{\varepsilon}}\langle w\rangle^{-l_0+\tilde{\varepsilon}}\, dw + \int_{B_R(0)} \tilde{C}_l\left(x+\langle\xi\rangle^{-\delta}w\right)\langle w\rangle^{-\tilde{\varepsilon}}\langle w\rangle^{-l_0+\tilde{\varepsilon}}\, dw \leq \varepsilon.
\end{aligned}$$

Using the previous estimate while verifying the norm-estimates of $\left|I_i^\theta(x,\xi)\right|$ for all $i \in \{1,2,3\}$ we obtain $C(x) \xrightarrow{|x|\to\infty} 0$ in the inequality (3.38).

If l_0 is odd, we can proof the claim in the same way, using Remark 3.11. □

The previous results enable us to show Theorem 3.10, now:

Proof of Theorem 3.10. We prove the claim in several steps: First we verify (3.30) in the case $|\beta| = 0$. Then we show (3.29) in the case $|\beta| = 0$ and $\partial_\xi^\alpha D_x^\beta a_L^\theta \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$. Afterwards on can use the cases $|\beta| = 0$ in order to verify (3.30) and (3.29) in the general case, which concludes the theorem. We obtain all those results by means of Proposition 3.13 and Proposition 3.14, which are modifications of the proofs of Proposition 4.8 and Proposition 4.6 in [2]. To this end we need to modify the analogous results of [2, Section 4.2] as already done in the proofs of Proposition 3.13 and Proposition 3.14. Note, that the generalized properties of the oscillatory integrals of Subsection 2.1 are needed for the proofs. The details are left to the reader. □

4 | FREDHOLM PROPERTY OF NON-SMOOTH PSEUDODIFFERENTIAL OPERATORS

The present section serves to show the main goal of this paper: The Fredholm property of non-smooth pseudodifferential operators fulfilling certain properties. For the proof of that statement we use the following compactness properties of non-smooth pseudodifferential operators verified by Marschall. They are special cases of Theorem 3 and Theorem 4 of [14].

Lemma 4.1. *Let $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, $M \in \mathbb{N}_0 \cup \{\infty\}$ with $M > \frac{n}{2}$. Moreover let $\tilde{m} \in \mathbb{N}_0$ and $0 < \tau < 1$ be such that $\tilde{m} + \tau > \frac{1-\rho}{1-\delta} \cdot \frac{n}{2}$ in case $\rho < 1$. Additionally let $a \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be such that*

$$\lim_{|x|+|\xi| \to \infty} (1 + |\xi|)^{-m} a(x, \xi) = 0.$$

Then for $(1 - \rho)\frac{n}{2} - (1 - \delta)(\tilde{m} + \tau) < s < \tilde{m} + \tau$

$$a(x, D_x) : H^{s+m}_2(\mathbb{R}^n) \to H^s_2(\mathbb{R}^n) \qquad \text{is compact.}$$

Lemma 4.2. *Let $m \in \mathbb{R}$, $0 \leq \delta \leq 1$, $1 < p < \infty$, $\tilde{m} \in \mathbb{N}_0$ and $0 < \tau < 1$. Moreover let $M \in \mathbb{N} \cup \{\infty\}$ with $M > n \cdot \max\left\{\frac{1}{2}, \frac{1}{p}\right\}$. Additionally let $a \in C^{\tilde{m},\tau} S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be such that*

$$\lim_{|x|+|\xi| \to \infty} (1 + |\xi|)^{-m} a(x, \xi) = 0.$$

Then for $-(1 - \delta)(\tilde{m} + \tau) < s < \tilde{m} + \tau$

$$a(x, D_x) : H^{s+m}_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n) \qquad \text{is compact.}$$

By means of those two lemmas we obtain the next two corollaries:

Proposition 4.3. *Let $0 \leq \delta \leq \rho \leq 1$, $m \in \mathbb{R}$, $M > \frac{n}{2}$ and $\varepsilon > 0$. Moreover let $\tilde{m} \in \mathbb{N}_0$ and $0 < \tau < 1$ be such that $\tilde{m} + \tau > \frac{1-\rho}{1-\delta} \cdot \frac{n}{2}$ if $\rho < 1$. Additionally let $a \in C^{\tilde{m},\tau} S^{m-\varepsilon}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap C^{\tilde{m},\tau} \dot{S}^{m-\varepsilon}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; 0)$. Then for all $s \in \mathbb{R}$ with*

$$(1 - \rho) \cdot \frac{n}{2} - (1 - \delta)(\tilde{m} + \tau) < s < \tilde{m} + \tau,$$

the operator

$$a(x, D_x) : H^{m+s}_2(\mathbb{R}^n) \to H^s_2(\mathbb{R}^n) \qquad \text{is compact.}$$

Proof. Since $a \in C^{\tilde{m},\tau} \dot{S}^{m-\varepsilon}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; 0)$ implies $|a(x, \xi)|\langle\xi\rangle^{-m} \xrightarrow{|x|+|\xi| \to \infty} 0$, the claim is a consequence of Lemma 4.1. □

Proposition 4.4. *Let $0 \leq \delta \leq 1$, $m \in \mathbb{R}$, $M > n \cdot \max\{\frac{1}{2}, \frac{1}{p}\}$ where $1 < p < \infty$ and $\varepsilon > 0$. Moreover let $\tilde{m} \in \mathbb{N}_0$ and $0 < \tau < 1$. Additionally let $a \in C^{\tilde{m},\tau} S^{m-\varepsilon}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap C^{\tilde{m},\tau} \dot{S}^{m-\varepsilon}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; 0)$. Then for all $s \in \mathbb{R}$ with*

$$-(1 - \delta)(\tilde{m} + \tau) < s < \tilde{m} + \tau,$$

the operator

$$a(x, D_x) : H^{m+s}_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n) \qquad \text{is compact.}$$

Proof. Since $a \in C^{\tilde{m},\tau} \dot{S}^{m-\varepsilon}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; 0)$ implies $|a(x, \xi)|\langle\xi\rangle^{-m} \xrightarrow{|x|+|\xi| \to \infty} 0$, the claim is a consequence of Lemma 4.2. □

In order to verify an asymptotic expansion of the product of two double symbols, we need the next theorem. It can be proved by means of the usual verifications of the similar result in the smooth case, see e.g. [11, Theorem 3.1]. For the convenience of the reader, we give a short sketch of the proof.

Theorem 4.5. *Let* $0 \le \delta \le \rho \le 1$, $m_1, m_2 \in \mathbb{R}$, $M_1, M_2 \in \mathbb{N}_0 \cup \{\infty\}$ *with* $M_1 > n+1$, $\tilde{m} \in \mathbb{N}_0$ *and* $0 < \tau < 1$. *For* $a \in C^{\tilde{m},\tau} S^{m_1,m_2}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1, M_2)$ *we define*

$$a_L(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \xi+\eta, x+y, \xi)\, dy\, đ\eta \quad \text{for all } x,\xi \in \mathbb{R}^n.$$

Additionally we set for all $\theta \in [0,1]$ *and* $\gamma \in \mathbb{N}_0^n$ *with* $|\gamma| \le M_1 - (n+1)$

$$r_{\gamma,\theta}(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta} \partial_\eta^\gamma D_y^\gamma a(x, \xi+\theta\eta, x+y, \xi)\, dy\, đ\eta \quad \text{for all } x,\xi \in \mathbb{R}^n.$$

Moreover we define $\tilde{M}_k := \min\{M_1 - k - (n+1); M_2\}$ *for all* $k \le M_1 - (n+1)$. *Then we get for all* $N \le M_1 - (n+1)$, *that*

$$a_L(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\eta^\alpha D_y^\alpha a(x, \xi+\eta, x+y, \xi)\Big|_{\eta=y=0} + R_N(x,\xi), \tag{4.1}$$

where

$$R_N(x,\xi) := N \cdot \sum_{|\gamma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} r_{\gamma,\theta}(x,\xi)\, d\theta \in C^{\tilde{m},\tau} S^{m_1+m_2-(\rho-\delta)\cdot N}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}_N)$$

and

$$\{r_{\gamma,\theta}(x,\xi) : |\theta| \le 1\} \subseteq C^{\tilde{m},\tau} S^{m_1+m_2-(\rho-\delta)\cdot N}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}_N) \quad \text{is bounded.}$$

If $\partial_\xi^\gamma D_y^\gamma a \in C^{\tilde{m},\tau} \hat{S}^{m_1-\rho, m_2+\delta}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M_1 - 1, M_2)$ *for* $|\gamma| = 1$ *then*

$$R_N(x,\xi) \in C^{\tilde{m},\tau} \dot{S}^{m_1+m_2-(\rho-\delta)\cdot N}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}_N)$$

for all $N \le M_1 - (n+1)$.

Proof. An application of the Taylor expansion formula to the second variable of a around ξ and integration by parts provides

$$a_L(x,\xi) = \sum_{|\gamma|<N} \frac{1}{\gamma!} \text{Os-}\iint e^{-iy\cdot\eta} D_y^\gamma \partial_\eta^\gamma a(x, \xi+\eta, x+y, \xi)\, dy\, đ\eta$$
$$+ N \sum_{|\gamma|=N} \text{Os-}\iint e^{-iy\cdot\eta} \frac{\eta^\gamma}{\gamma!} \int_0^1 (1-\theta)^{N-1} \partial_\eta^\gamma a(x, \xi+\theta\eta, x+y, \xi)\, d\theta\, dy\, đ\eta.$$

Next we need to exchange the oscillatory integral with the integral in the second term of the right side of the previous equality. Hence we choose an arbitrary $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$ and let $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = N$. Now let $l = n+1$ and $\tilde{l} = 1 + \lceil \frac{m_1+n}{1-\delta} \rceil$. Then we obtain due to the Theorem of Fubini and integration by parts using $e^{-iy\cdot\eta} = A^{\tilde{l}}(D_y, \eta) A^l(D_\eta, y) e^{-iy\cdot\eta}$, see (2.3) and (2.4) for the definition of $A^l(D_., .)$, for each $\varepsilon > 0$:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 e^{-iy\cdot\eta} \chi(\varepsilon y)\chi(\varepsilon\eta)\eta^\gamma (1-\theta)^{N-1} \partial_\eta^\gamma a(x, \xi+\theta\eta, x+y, \xi)\, d\theta\, dy\, đ\eta$$
$$= \int_0^1 (1-\theta)^{N-1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\eta} A^{\tilde{l}}(D_y, \eta) A^l(D_\eta, y) \Big\{ \chi(\varepsilon\eta) D_y^\gamma \Big[\chi(\varepsilon y) \partial_\eta^\gamma a(x, \xi+\theta\eta, x+y, \xi) \Big] \Big\}\, dy\, đ\eta\, d\theta. \tag{4.2}$$

Here the assumptions of the Theorem of Fubini and of integration by parts can be verified. Since $\chi \in \mathcal{S}(\mathbb{R}^n)$, $D_y^\alpha \chi(\varepsilon y) \to 0$ for $\varepsilon \to 0$ if $|\alpha| \ne 0$. Hence we get by interchanging the limit and the integration on account of (4.2) and since the integrand has an

L^1-majorant:

$$\text{Os -}\iint e^{-iy\cdot\eta}\frac{\eta^\gamma}{\gamma!}\int_0^1 (1-\theta)^{N-1}\partial_\eta^\gamma a\big(x,\xi+\theta\eta,x+y,\xi\big)\,d\theta\,dy\,đ\eta$$

$$=\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\eta}A^{\tilde{l}}\big(D_y,\eta\big)A^l\big(D_\eta,y\big)\Big\{D_y^\gamma\partial_\eta^\gamma a\big(x,\xi+\theta\eta,x+y,\xi\big)\Big\}\,dy\,đ\eta\,d\theta$$

$$=\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!}\text{Os -}\iint e^{-iy\cdot\eta}D_y^\gamma\partial_\eta^\gamma a\big(x,\xi+\theta\eta,x+y,\xi\big)\,dy\,đ\eta\,d\theta,$$

where the last equality holds because of Theorem 2.5. Hence (4.1) holds. The rest of the claim is a consequence of Theorem 3.10. □

As a consequence of the previous theorem, we obtain:

Corollary 4.6. *Let $\tilde{m}_1\in\mathbb{N}_0$, $0<\tau_1<1$, $m_1,m_2\in\mathbb{R}$, $0\le\delta<\rho\le 1$; $M_1,M_2\in\mathbb{N}_0\cup\{\infty\}$ with $M_1>n+1$. Additionally let $N:=M_1-(n+1)$. For $a_1\in C^{\tilde{m}_1,\tau_1}S^{m_1}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;M_1\big)$ and $a_2\in S^{m_2}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;M_2\big)$ we define*

$$a(x,\xi):=Os\text{ -}\iint e^{-iy\cdot\eta}a_1\big(x,\xi+\eta\big)a_2(x+y,\xi)\,dy\,đ\eta$$

and for all $k\in\mathbb{N}$ with $k\le N$, $\gamma\in\mathbb{N}_0^n$ with $|\gamma|=N$ and $\theta\in[0,1]$ we set

- $a_1\sharp_k a_2(x,\xi):=\sum_{|\gamma|<k}\frac{1}{\gamma!}\partial_\xi^\gamma a_1(x,\xi)D_x^\gamma a_2(x,\xi),$
- $r_{\gamma,\theta}(x,\xi):=Os\text{ -}\iint e^{-iy\cdot\eta}\partial_\eta^\gamma a_1\big(x,\xi+\theta\eta\big)D_y^\gamma a_2(x+y,\xi)\,dy\,đ\eta$

for all $x,\xi\in\mathbb{R}^n$. Moreover we define $R_k:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ as in Theorem 4.5. Then

$$a(x,\xi)=a_1\sharp_k a_2(x,\xi)+R_k(x,\xi)\qquad\text{for all } x,\xi\in\mathbb{R}^n$$

and with $\tilde{M}_k:=\min\big\{M_1-k+1;M_2\big\}$ and $\tilde{N}_k:=\min\big\{M_1-k-(n+1);M_2\big\}$ we obtain

- $a_1\sharp_k a_2(x,\xi)\in C^{\tilde{m}_1,\tau_1}S^{m_1+m_2}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}_k\big),$
- $R_k(x,\xi)\in C^{\tilde{m}_1,\tau_1}S^{m_1+m_2-(\rho-\delta)k}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;\tilde{N}_k\big).$

In particular we have $a(x,\xi)\in C^{\tilde{m}_1,\tau_1}S^{m_1+m_2}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;\tilde{N}_1\big)$. If we even have $a_2\in\tilde{S}^{m_2}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;M_2\big)$ or $D_x^\beta a_2\in\dot{S}^{m_2}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;M_2\big)$ for all $\beta\in\mathbb{N}_0^n$ with $|\beta|=1$, then $R_k(x,\xi)\in C^{\tilde{m}_1,\tau_1}\dot{S}^{m_1+m_2-(\rho-\delta)k}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;\tilde{N}_k\big)$ for all $k\in\mathbb{N}$ with $k\le N$.

Proof. Since $a_1(x,\xi)a_2(y,\xi')\in C^{\tilde{m}_1,\tau_1}S^{m_1,m_2}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n;M_1,M_2\big)$ we just need to show $a_1\sharp_k a_2(x,\xi)\in C^{\tilde{m}_1,\tau_1}S^{m_1+m_2}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}_k\big)$, the rest is a consequence of Theorem 4.5. Let $k\in\mathbb{N}$ with $k\le N$ be arbitrary and $\alpha,\beta,\gamma\in\mathbb{N}_0^n$ with $|\gamma|<k$, $|\beta|\le\tilde{m}_1$ and $|\alpha|\le\tilde{M}_k$. The choice of a_1 and a_2 provides by means of the Leibniz rule

$$\Big|\partial_\xi^\alpha D_x^\beta\Big\{\partial_\xi^\gamma a_1(x,\xi)D_x^\gamma a_2(x,\xi)\Big\}\Big|\le C_{\alpha,\beta,\gamma}(x)\langle\xi\rangle^{m_1+m_2-(\rho-\delta)|\gamma|-\rho|\alpha|+\delta|\beta|}\tag{4.3}$$

for all $x,\xi\in\mathbb{R}^n$, where $C_{\alpha,\beta,\gamma}(x)$ is bounded. On account of (3.1) we know, that $D_x^\gamma a_2(x,\xi)\in C^{\tilde{m}_1,\tau_1}S^{m_2+\delta|\gamma|}_{\rho,\delta}\big(\mathbb{R}^n\times\mathbb{R}^n;M_2\big)$. Hence an application of Lemma 2.1, Remark 3.1 and the Leibniz rule provides

$$\Big\|\partial_\xi^\alpha\Big\{\partial_\xi^\gamma a_1(x,\xi)D_x^\gamma a_2(x,\xi)\Big\}\Big\|_{C^{\tilde{m}_1,\tau_1}(\mathbb{R}^n_x)}\le C_{\alpha,\tilde{m}_1,\gamma}\langle\xi\rangle^{m_1+m_2-(\rho-\delta)|\gamma|-\rho|\alpha|+\delta(\tilde{m}_1+\tau_1)}.\tag{4.4}$$

A combination of (4.3) and (4.4) yields

$$\partial_\xi^\gamma a_1(x,\xi) D_x^\gamma a_2(x,\xi) \in C^{\tilde{m}_1,\tau_1} S^{m_1+m_2-(\rho-\delta)|\gamma|}\big(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}_k\big)$$
$$\subseteq C^{\tilde{m}_1,\tau_1} S^{m_1+m_2}\big(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}_k\big).$$

Hence $a_1 \sharp_k a_2(x,\xi) \in C^{\tilde{m}_1,\tau_1} S^{m_1+m_2}_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}_k\big)$. □

With the previous corollary at hand, we now can show the next statement:

Theorem 4.7. *Let $\tilde{m}_1 \in \mathbb{N}_0$, $\tilde{m}_2 \in \mathbb{N}$, $0 < \tau_1, \tau_2 < 1$, $m_1, m_2 \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$. Furthermore let $p = 2$ if $\rho \neq 1$ and $1 < p < \infty$ else. We choose a $\theta \notin \mathbb{N}_0$ with $\theta \in \big(0, (\tilde{m}_2+\tau_2)(\rho-\delta)\big)$, $\tilde{\varepsilon} \in \big(0, \min\big\{(\rho-\delta)\tau_2; (\rho-\delta)\big(\tilde{m}_2+\tau_2\big)-\theta;\theta\big)\big\}\big)$ and define $(\tilde{m},\tau) := (\lfloor s \rfloor, s - \lfloor s \rfloor)$, where $s := \min\{\tilde{m}_1+\tau_1; \tilde{m}_2+\tau_2-\lfloor\theta\rfloor\}$. Additionally let $M_1, M_2 \in \mathbb{N}_0 \cup \{\infty\}$ with $M_1 > (n+1) + \lceil\theta\rceil + n\max\left\{\frac{1}{2},\frac{1}{p}\right\}$ and $M_2 > n \cdot \max\left\{\frac{1}{2},\frac{1}{p}\right\}$. Moreover let $a_1 \in C^{\tilde{m}_1,\tau_1} S^{m_1}_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M_1\big)$ and $a_2 \in C^{\tilde{m}_2,\tau_2} \tilde{S}^{m_2}_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M_2\big)$ such that*

$$a_2(x,\xi) \xrightarrow{|x|\to\infty} a_2(\infty,\xi) \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Then we get for each $s \in \mathbb{R}$ fulfilling $(1-\rho)\frac{n}{2} - (1-\delta)(\tilde{m}_2+\tau_2) + \theta + \tilde{\varepsilon} < s + m_1 < \tilde{m} + \tau_2$ and $(1-\rho)\frac{n}{2} - (1-\delta)(\tilde{m}+\tau) + \frac{\tilde{m}+\tau}{\tilde{m}_2+\tau_2}(\theta+\tilde{\varepsilon}) < s < \tilde{m}+\tau$, that

$$a_1\big(x,D_x\big)a_2\big(x,D_x\big) - \big(a_1 \sharp_{\lceil\theta\rceil} a_2\big)\big(x,D_x\big) : H_p^{s+m_1+m_2}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n) \quad \text{is compact.}$$

where $a_1 \sharp_{\lceil\theta\rceil} a_2(x,\xi)$ is defined as in Corollary 4.6.

Remark 4.8. If we weaken the condition for the second symbol in the previous theorem to $a_2 \in C^{\tilde{m}_2,\tau_2} S^{m_2}_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M_2\big)$, we can show in the same way as in the proof of Theorem 4.7, the compactness of

$$a_1\big(x,D_x\big)a_2\big(x,D_x\big) - \big(a_1 \sharp_{\lceil\theta\rceil} a_2\big)\big(x,D_x\big) : H_p^{s+m_1+m_2-\varepsilon}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$$

for some $\varepsilon > 0$.

Proof of Theorem 4.7. Let $1 < p < \infty$ if $\rho = 1$ and $p = 2$ else. Setting $\gamma := \delta + \frac{\theta+\tilde{\varepsilon}}{\tau_2+\tilde{m}_2}$ Corollary 4.6 provides for $k \in \mathbb{N}$ with $k \le M_1 - (n+1)$ and $\tilde{M}_k := \min\big\{M_1 - k + 1; M_2\big\}$ that the symbol $a_1 \sharp_k a_2$ has the following properties if $a_2 \in \tilde{S}^{m_2}_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M_2\big)$:

i) $a_1 \sharp_k a_2 \in C^{\tilde{m}_1,\tau_1} S^{m_1+m_2}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M}_k\big)$,
ii) $\sigma\big(a_1\big(x,D_x\big)a_2\big(x,D_x\big)\big) - a_1 \sharp_k a_2 \in C^{\tilde{m}_1,\tau_1} \dot{S}^{m_1+m_2-(\rho-\delta)\cdot k}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N}_k\big)$,

where $\tilde{N}_k := \min\big\{M_1 - k - (n+1); M_2\big\}$ and

$$\sigma\big(a_1\big(x,D_x\big)a_2\big(x,D_x\big)\big) := \text{Os-}\iint e^{-iy\cdot\eta} a_1\big(x,\xi+\eta\big)a_2\big(x+y,\eta\big)\, dy\, đ\eta.$$

Now let $a_2 \in C^{\tilde{m}_2,\tau_2} \tilde{S}^{m_2}_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M_2\big)$ be arbitrary. By means of Lemma 3.8 and Lemma 3.7 we get for all $\beta \in \mathbb{N}_0^n$ with $0 < |\beta| \le \tilde{m}_2$

iii) $a_2^b \in C^{\tilde{m}_2,\tau_2} \tilde{S}^{m_2-\theta}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; M_2\big) \cap C^{\tilde{m}_2,\tau_2} \dot{S}^{m_2-\theta}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; 0)$,
iv) $a_2^\sharp \in S^{m_2}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; M_2\big)$ and $D_x^\beta a_2^\sharp \in \dot{S}^{m_2}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; M_2\big)$,
v) $a_2(x,\xi) = a_2^b(x,\xi) + a_2^\sharp(x,\xi)$ for all $x, \xi \in \mathbb{R}^n$,

Now let s be as in the assumptions. Due to Proposition 4.4 and Proposition 4.3 we know that

$$a_2^b\big(x,D_x\big) : H_p^{s+m_1+m_2}(\mathbb{R}^n) \to H_p^{s+m_1}(\mathbb{R}^n) \qquad \text{is compact.}$$

On account of the boundedness of $a_1(x, D_x) : H_p^{s+m_1}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$, see Theorem 3.2, we obtain

$$a_1(x, D_x)a_2^b(x, D_x) : H_p^{s+m_1+m_2}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n) \qquad \text{is compact.} \tag{4.5}$$

Then we obtain by means of the Leibniz rule, Lemma 2.1 and $a_2^b \in C^{\tilde{m}_2,\tau_2}\dot{S}^{m_2-\theta}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; 0)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < \lceil \theta \rceil$:

$$\partial_\xi^\alpha a_1(x,\xi) D_x^\alpha a_2^b(x,\xi) \in C^{\tilde{m},\tau} S^{m_1+m_2-\theta}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; \min\{M_1 - |\alpha|; M_2\}\big) \cap C^{\tilde{m},\tau}\dot{S}^{m_1+m_2-\theta}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; 0). \tag{4.6}$$

Due to (4.6), Proposition 4.4 and Proposition 4.3 provides for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < \lceil \theta \rceil$:

$$\Big(\partial_\xi^\alpha a_1 D_x^\alpha a_2^b\Big)(x, D_x) : H_p^{s+m_1+m_2}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n) \qquad \text{is compact.} \tag{4.7}$$

Since $a_1 \in C^{\tilde{m}_1,\tau_1} S^{m_1}_{\rho,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; M_1)$ and (iv) holds, we obtain together with (v) and (i), (ii) applied on $a_2^\sharp$ instead on a_2

$$\begin{aligned} &a_1(x, D_x)a_2(x, D_x) - (a_1 \sharp_{\lceil\theta\rceil} a_2)(x, D_x) \\ &= a_1(x, D_x)a_2^b(x, D_x) - \sum_{|\alpha|<\lceil\theta\rceil} \frac{1}{\alpha!}\big(\partial_\xi^\alpha a_1 D_x^\alpha a_2^b\big)(x, D_x) + R_{\lceil\theta\rceil}(x, D_x), \end{aligned} \tag{4.8}$$

where

$$R_{\lceil\theta\rceil}(x,\xi) \in C^{\tilde{m}_1,\tau_1}\dot{S}^{m_1+m_2-(\rho-\delta)\lceil\theta\rceil}_{\rho,\gamma}\big(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N}_{\lceil\theta\rceil}\big).$$

Because of Proposition 4.4 and Proposition 4.3, we get

$$R_{\lceil\theta\rceil}(x, D_x) : H_p^{s+m_1+m_2}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n) \qquad \text{is compact.} \tag{4.9}$$

A combination of (4.8), (4.5), (4.7) and (4.9) yields the claim. □

In order to verify the main result of our paper, we use:

Lemma 4.9. *Let $\tilde{m} \in \mathbb{N}_0$, $N \in \mathbb{N}$, $0 < \tau < 1$, $0 \le \delta < \rho \le 1$ and $M \in \mathbb{N}_0 \cup \{\infty\}$. Let $a \in C^{\tilde{m},\tau}\tilde{S}^0_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}(\mathbb{C}^N)\big)$ be such that property 1) of Theorem 1.1 hold. Moreover let $\psi \in C_b^\infty(\mathbb{R}^n)$ be such that $\psi(x) = 0$ if $|x| \le 1$ and $\psi(x) = 1$ if $|x| \ge 2$. Then $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^{N\times N}$ defined by*

$$b(x,\xi) := \psi\big(R^{-2}(|x|^2 + |\xi|^2)\big)a(x,\xi)^{-1} \qquad \text{for all } x, \xi \in \mathbb{R}^n$$

is an element of $C^{\tilde{m},\tau}\tilde{S}^0_{\rho,\delta}\big(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}(\mathbb{C}^N)\big)$.

Proof. First we assume that $N = 1$. We remark that $b(x, \xi)$ is 0 if $|x|^2 + |\xi|^2 \le R^2$ and $b(x, \xi) = 1$, if $|x|^2 + |\xi|^2 \ge 2R^2$. Using property 1) of a we can verify

$$\big\| a(.,\xi)^{-1} \big\|_{C^0(\mathbb{R}^n)} \le C \qquad \text{and} \qquad \big\| a(.,\xi)^{-1} \big\|_{C^{0,\tau}(\mathbb{R}^n)} \le C \tag{4.10}$$

for all $|\xi| \ge R$. Due to the product rule we can write each derivative $\partial_\xi^\alpha D_x^\beta a(x,\xi)^{-1}$ ($\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \le M$, $|\beta| \le \tilde{m}$) as the sum of terms of the form

$$\partial_\xi^{\alpha_1} D_x^{\beta_1} a(x,\xi) \cdot \ldots \cdot \partial_\xi^{\alpha_k} D_x^{\beta_k} a(x,\xi) \cdot a(x,\xi)^{-l},$$

where $\alpha_1 + \cdots + \alpha_k = \alpha$ and $\beta_1 + \cdots + \beta_k = \beta \in \mathbb{N}_0^n$, $k, l \in \mathbb{N}$. By means of Lemma 2.1, the inequality (4.10), property 1) and $a \in C^{\tilde{m},\tau}\tilde{S}^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we get

$$\Big\| \partial_\xi^{\alpha_1} D_x^{\beta_1} a(x,\xi) \cdot \ldots \cdot \partial_\xi^{\alpha_k} D_x^{\beta_k} a(x,\xi) \cdot a(x,\xi)^{-l} \Big\|_{C^{0,\tau}(\mathbb{R}^n_x)} \le C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta(|\beta|+\tau)},$$

$$\Big| \partial_\xi^{\alpha_1} D_x^{\beta_1} a(x,\xi) \cdot \ldots \cdot \partial_\xi^{\alpha_k} D_x^{\beta_k} a(x,\xi) \cdot a(x,\xi)^{-l} \Big| \le C_{\alpha,\beta}(x)\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}$$

for all $x, \xi \in \mathbb{R}^n$ with $|\xi| \geq R$. Here $C_{\alpha,\beta}(x)$ is bounded and $C_{\alpha,\beta}(x) \xrightarrow{|x|\to\infty} 0$ if $|\beta| \neq 0$. Hence we obtain for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $|\beta| \leq \tilde{m}$:

$$\left\| \partial_\xi^\alpha a(x,\xi)^{-1} \right\|_{C^{\tilde{m},\tau}(\mathbb{R}^n_x)} \leq C_{\alpha,\tilde{m}} \langle \xi \rangle^{-\rho|\alpha|+\delta(\tilde{m}+\tau)} \quad \text{for all } \xi \in \mathbb{R}^n \text{ with } |\xi| \geq R, \tag{4.11}$$

$$\left| \partial_\xi^\alpha D_x^\beta a(x,\xi)^{-1} \right| \leq C_{\alpha,\beta}(x) \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|} \quad \text{for all } x, \xi \in \mathbb{R}^n \text{ with } |x|^2 + |\xi|^2 \geq R^2. \tag{4.12}$$

Here $C_{\alpha,\beta}(x)$ is bounded and $C_{\alpha,\beta}(x) \xrightarrow{|x|\to\infty} 0$ if $|\beta| \neq 0$. Now let $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $|\beta| \leq \tilde{m}$ be arbitrary. On account of the product rule and the definition of ψ, we obtain

$$\left| \partial_\xi^\alpha D_x^\beta b(x,\xi) \right| = 0 \qquad \text{for all } x, \xi \in \mathbb{R}^n \text{ with } |x|^2 + |\xi|^2 \leq R^2. \tag{4.13}$$

Now let $\xi \in \mathbb{R}^n$ with $0 \leq |\xi|^2 \leq 2R^2$. Then we have for all $\alpha_1, \beta_1 \in \mathbb{N}_0^n$, that $\langle \xi \rangle^{\rho|\alpha_1|-\delta|\beta_1|} \leq C_R$. Together with (4.11) and (4.12) an application of the product rule and Lemma 2.1 provides

$$\left\| \partial_\xi^\alpha D_x^\beta b(x,\xi) \right\|_{C^{0,\tau}(\mathbb{R}^n_x)} \leq C_{\alpha,\beta,R} \langle \xi \rangle^{-\rho|\alpha|+\delta(|\beta|+\tau)}, \tag{4.14}$$

where $C_{\alpha,\beta,R}$ is independent of $\xi \in \mathbb{R}^n$ with $0 \leq |\xi|^2 \leq 2R^2$. Moreover we obtain for all $x, \xi \in \mathbb{R}^n$ with $R^2 \leq |x|^2 + |\xi|^2 \leq 2R^2$:

$$\begin{aligned} \left| \partial_\xi^\alpha D_x^\beta b(x,\xi) \right| &\leq \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C_{\alpha_1,\beta_1} \left| \partial_\xi^{\alpha_1} D_x^{\beta_1} \psi\left(R^{-2}\left(|x|^2+|\xi|^2\right)\right) \right| \left| \partial_\xi^{\alpha_2} D_x^{\beta_2} a(x,\xi)^{-1} \right| \\ &\leq C_{\alpha,\beta,R}(x) \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|}, \end{aligned} \tag{4.15}$$

where $C_{\alpha,\beta,R}(x)$ is independent of $\xi \in \mathbb{R}^n$ with $R^2 \leq |\xi|^2 \leq 2R^2$ and bounded with respect to x. Now let $\xi \in \mathbb{R}^n$ with $|\xi|^2 \geq 2R^2$. Then $\psi\left(R^{-2}\left(|x|^2+|\xi|^2\right)\right) = 1$. Hence we obtain by means of (4.11)

$$\left\| \partial_\xi^\alpha D_x^\beta b(x,\xi) \right\|_{C^{0,\tau}(\mathbb{R}^n_x)} \leq C_{\alpha,\beta,R} \langle \xi \rangle^{-\rho|\alpha|+\delta(|\beta|+\tau)}, \tag{4.16}$$

where $C_{\alpha,\beta,R}$ is independent of $\xi \in \mathbb{R}^n$ with $|\xi|^2 \geq 2R^2$. Moreover (4.12) implies for all $x, \xi \in \mathbb{R}^n$ with $|x|^2 + |\xi|^2 \geq 2R^2$

$$\left| \partial_\xi^\alpha D_x^\beta b(x,\xi) \right| = \left| \partial_\xi^\alpha D_x^\beta a(x,\xi)^{-1} \right| \leq C_{\alpha,\beta,R}(x) \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|}, \tag{4.17}$$

where $C_{\alpha,\beta,R}(x)$ is bounded, independent of $\xi \in \mathbb{R}^n$ with $|\xi|^2 \geq 2R^2$ and $C_{\alpha,\beta,R}(x) \xrightarrow{|x|\to\infty} 0$ if $|\beta| \neq 0$. Now a combination of (4.13), (4.14), (4.15), (4.16) and (4.17) provides the claim: For all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$, $|\beta| \leq \tilde{m}$ we have

$$\left\| \partial_\xi^\alpha b(x,\xi) \right\|_{C^{\tilde{m},\tau}(\mathbb{R}^n_x)} = \max_{|\gamma| \leq \tilde{m}} \left\| \partial_\xi^\alpha D_x^\gamma b(x,\xi) \right\|_{C^{0,\tau}(\mathbb{R}^n_x)} \leq C_{\alpha,\tilde{m},R} \langle \xi \rangle^{-\rho|\alpha|+\delta(\tilde{m}+\tau)}$$

for all $\xi \in \mathbb{R}^n$ and

$$\left| \partial_\xi^\alpha D_x^\gamma b(x,\xi) \right| \leq C_{\alpha,\tilde{m},R}(x) \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|} \qquad \text{for all } x, \xi \in \mathbb{R}^n,$$

where $C_{\alpha,\tilde{m},R}(x)$ is bounded and $C_{\alpha,\beta,R}(x) \xrightarrow{|x|\to\infty} 0$ if $|\beta| \neq 0$.

Finally, let us consider the general case $N \in \mathbb{N}$. Then the case $N = 1$ implies that $\tilde{b}$ defined by

$$\tilde{b}(x,\xi) := \psi\left(R^{-2}\left(|x|^2+|\xi|^2\right)\right) \det(a(x,\xi))^{-1} \quad \text{for all } x, \xi \in \mathbb{R}^n$$

is an element of $C^{\tilde{m},\tau} \tilde{S}^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Now the statement of the lemma easily follows from Cramer's rule and the fact that $C^{\tilde{m},\tau} \tilde{S}^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is closed with respect to pointwise multiplication. □

Using the main idea of the analog result in the smooth case, see [11, Theorem 5.16], we now are able to verify Theorem 1.1:

Proof of Theorem 1.1. First of all we assume, that $m = 0$. In order to prove the claim let us choose $\psi \in C_b^\infty(\mathbb{R}^n)$ such that $\psi(x) = 0$ if $|x| \leq 1$ and $\psi(x) = 1$ if $|x| \geq 2$. Then $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathscr{L}\left(\mathbb{C}^N\right)$ defined by

$$b(x,\xi) := \psi\left(R^{-2}\left(|x|^2 + |\xi|^2\right)\right)a(x,\xi)^{-1} \qquad \text{for all } x, \xi \in \mathbb{R}^n$$

is an element of $C^{\tilde{m},\tau}\tilde{S}^0_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\left(\mathbb{C}^N\right)\right)$ on account of Lemma 4.9. Using Theorem 4.7 we obtain for all $s \in \mathbb{R}$ with $(1-\rho)\frac{n}{2} - (1-\delta)(\tilde{m}+\tau) + \theta + \tilde{\varepsilon} < s < \tilde{m} + \tau$ and $1 < p < \infty$ with $p = 2$ if $\rho \neq 1$:

i) $a\left(x, D_x\right)b\left(x, D_x\right) = OP(ab) + R_1$,
ii) $b\left(x, D_x\right)a\left(x, D_x\right) = OP(ab) + R_2$,

where

$$R_1, R_2 : H_p^s(\mathbb{R}^n)^N \to H_p^s(\mathbb{R}^n)^N \qquad \text{are compact.}$$

By means of the Leibniz formula and Lemma 2.1 we get

$$a(x,\xi)b(x,\xi) - I \in C^{\tilde{m},\tau}S^0_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\left(\mathbb{C}^N\right)\right).$$

An application of Lemma 4.1 in the case $\rho \neq 1$ and Lemma 4.2 else provides, that

$$OP(ab - I) : H_p^s(\mathbb{R}^n)^N \to H_p^s(\mathbb{R}^n)^N \qquad \text{is compact} \tag{4.18}$$

for all $(1-\delta)\frac{n}{2} - (1-\delta)(\tilde{m}+\tau) + \theta + \tilde{\varepsilon} < s < \tilde{m} + \tau$, where $p = 2$ if $\rho \neq 1$. Together with i) we obtain:

$$a\left(x, D_x\right)b\left(x, D_x\right) = OP(ab) - Id + Id + R_1 = Id + \left[OP(ab - I) + R_1\right],$$

where

$$OP(ab - I) + R_1 : H_p^s(\mathbb{R}^n)^N \to H_p^s(\mathbb{R}^n)^N \qquad \text{is compact}$$

for all $(1-\delta)\frac{n}{2} - (1-\delta)(\tilde{m}+\tau) + \theta + \tilde{\varepsilon} < s < \tilde{m} + \tau$, where $p = 2$ if $\rho \neq 1$. Analogous we obtain on account of ii) and (4.18)

$$b\left(x, D_x\right)a\left(x, D_x\right) = OP(ab) - Id + Id + R_2 = Id + \left[OP(ab - 1) + R_2\right],$$

where

$$OP(ab - I) + R_2 : H_p^s(\mathbb{R}^n)^N \to H_p^s(\mathbb{R}^n)^N \qquad \text{is compact}$$

for all $(1-\delta)\frac{n}{2} - (1-\delta)(\tilde{m}+\tau) + \theta + \tilde{\varepsilon} < s < \tilde{m} + \tau$, where $p = 2$ if $\rho \neq 1$. This implies the claim for $m = 0$. For general $m \in \mathbb{R}$, we use that $\langle D_x \rangle^m : H_p^{m+s}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$ is a Fredholm operator for all $s \in \mathbb{R}$ since it is invertible. An application of the case $m = 0$ to

$$\tilde{a}(x,\xi) := a(x,\xi)\langle\xi\rangle^{-m} \in C^{\tilde{m},\tau}\tilde{S}^0_{\rho,\delta}\left(\mathbb{R}^n \times \mathbb{R}^n; M; \mathscr{L}\left(\mathbb{C}^N\right)\right)$$

yields that $\tilde{a}\left(x, D_x\right) : H_p^s(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$ is a Fredholm operator. Since the composition of two Fredholm operators is a Fredholm operator again, we finally obtain the statement of this theorem on account of

$$a\left(x, D_x\right) = \tilde{a}(x, D_x)\operatorname{diag}\left(\langle D_x\rangle^m, \dots, \langle D_x\rangle^m\right) : H_p^{m+s}(\mathbb{R}^n)^N \to H_p^s(\mathbb{R}^n)^N,$$

where $\operatorname{diag}\left(\langle D_x\rangle^m, \dots, \langle D_x\rangle^m\right)$ is the $N \times N$ diagonal operator matrix with diagonal entries $\langle D_x\rangle^m$. □

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